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Twisted Fourier–Mukai transforms for holomorphic symplectic four-folds

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Abstract

We apply the methods of Căldăraru to construct a twisted Fourier–Mukai transform between a pair of holomorphic symplectic four-folds which are fibred by Lagrangian abelian surfaces. More precisely, we obtain an equivalence between the derived category of coherent sheaves on a certain Lagrangian fibration and the derived category of twisted sheaves on its 'mirror' partner. As a corollary, we extend the original Fourier–Mukai transform to degenerations of abelian surfaces. Another consequence of the general theory is that the holomorphic symplectic four-fold and its mirror are connected by a one-parameter family of deformations through Lagrangian fibrations.

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1. Introduction

Matsushita [37] proved that a projective holomorphic symplectic manifold can only be fibred by (holomorphic) Lagrangian abelian varieties; his results in [37,38] also suggest that the base of the fibration must be projective space. In [45] the author reviewed what is known about such fibrations, and speculated on what may be true. In particular, we hope to obtain a classification (up to deformation) of holomorphic symplectic manifolds via this approach.

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A central problem is: *can a Lagrangian fibration which does not admit a global section be deformed to one that does?* This is the motivation behind the present paper, which answers the question affirmatively for a particular example (Theorem 25) while introducing ideas which should have wider applications.

We investigate holomorphic symplectic four-folds fibred by Lagrangian abelian surfaces. Following Altman and Kleiman [3], we associate to such a Lagrangian fibration $X \to B$ its compactified relative Picard scheme $P := \overline{\mathrm{Pic}}^0(X/B)$, which is fibred over B and admits a section. We regard P as the dual fibration, and the double-dual fibration is $X^0 := \overline{\mathrm{Pic}}^0(P/B)$. If the singular fibres of X are not too bad, both P and X^0 are well-defined and smooth. It can also happen that X and X^0 are locally isomorphic as fibrations, even around singular fibres, and then X is a torsor over X^0 (in this article, we use 'torsor' to mean a compactified principal homogeneous space). Our goal is to study the relation between X and X^0. From the point of view of classification of holomorphic symplectic manifolds, any deformation between X and X^0 would suffice; but it is natural to look for a deformation via Lagrangian fibrations and this is the approach we shall take.

The analogous situation for elliptic surfaces was studied by Kodaira [29,30]; for higher dimensional elliptic fibred varieties it is known as Ogg–Shafarevich theory (see [15], for instance). In those cases we use the compactified relative Jacobian $J := \overline{\mathrm{Jac}}(X/B)$ of an elliptic fibration $X \to B$ (since elliptic curves are self-dual, we do not need to take the double-dual fibration). The group classifying all torsors X over J is the (cohomological) analytic Brauer group $\mathrm{H}^2(J, \mathcal{O}^*)$ of J. Recently Căldăraru [13] (see also [11] and [12]) gave a more conceptual explanation for the appearance of the Brauer group in this context: J can be interpreted as a moduli space of sheaves on X, and there is a holomorphic gerbe $\beta \in \mathrm{H}^2(J, \mathcal{O}^*)$ obstructing the existence of a universal sheaf for the moduli problem. Then X may be regarded as J 'twisted' by β. Căldăraru also defined a twisted Fourier–Mukai transform: an equivalence between the derived category of coherent sheaves on X and the derived category of β^{-1}-twisted sheaves on J. The existence of such an equivalence is indicative of the close geometric relation between the spaces X and J.

Among the examples considered by Căldăraru are elliptic K3 surfaces and elliptic Calabi–Yau three-folds. In this paper we construct a twisted Fourier–Mukai transform for holomorphic symplectic four-folds fibred by Lagrangian abelian surfaces (Theorem 23). The compactified relative Picard scheme P will be, a priori, a moduli space of sheaves on the four-fold X. We will show that it is smooth and holomorphic symplectic, by identifying it with another well-known holomorphic symplectic four-fold. The double-dual fibration X^0 will also be a smooth holomorphic symplectic four-fold. Moreover, X will be a torsor over X^0, and will therefore correspond to a gerbe in $\mathrm{H}^2(P, \mathcal{O}^*)$. By analyzing the space $\mathrm{H}^2(P, \mathcal{O}^*)$ of gerbes on P, we will show that there is a one-parameter family of Lagrangian fibrations connecting X, which does not admit a global section, to X^0, which does (Theorem 25). It should be stressed that our main interest is in obtaining geometric results such as this, and the application of the more abstract theory (of moduli spaces, gerbes, obstructions to universal sheaves, etc.) is just a tool to help understand the structure of the fibrations.

The twisted Fourier–Mukai transform will be an equivalence between the derived category of X and the twisted derived category of P. It can be regarded as a relative version of the Fourier–Mukai transform between an abelian surface and its dual. The twist arises when one tries to 'assemble' the fibrewise transforms into a global transform. As a corollary, we extend the Fourier–Mukai transform for abelian surfaces to degenerations of abelian surfaces (Corollary 21).

It is widely believed that Fourier–Mukai transforms are related to homological mirror symmetry, although the precise relation is not yet fully understood. On hyperkähler manifolds, a holomorphic Lagrangian fibration becomes a special Lagrangian fibration after a rotation of complex structures. We expect that our equivalence is a manifestation of homological mirror symmetry in the presence of a B-field, as applied to SYZ fibrations [49]. In this setting, a B-field is a lift of the gerbe to $\mathrm{H}^2(P,\mathbb{Q})$, i.e., it is a class $B \in \mathrm{H}^2(P,\mathbb{Q})$ whose $(0,2)$-part $B^{0,2} \in \mathrm{H}^2(P,\mathcal{O})$ maps to $\beta \in \mathrm{H}^2(P,\mathcal{O}^*)$ under the exponential map (see Huybrechts and Stellari [27]). Similar examples of mirror pairs of hyperkähler manifolds involving B-fields were constructed by Hausel and Thaddeus [22], though their examples are non-compact, being based on the Hitchin system. We expect that our methods will produce more compact examples in higher dimensions, and that these will be related to Hausel and Thaddeus' examples via the construction of Donagi, Ein, and Lazarsfeld [16] (namely, the compactified Hitchin system is a degeneration of the Beauville–Mukai system).

The paper is organized as follows. In Section 2 we review results of Mukai, Bridgeland, Maciocia, and Căldăraru on Fourier–Mukai and twisted Fourier–Mukai transforms. In Section 3 we introduce a pair of holomorphic symplectic four-folds which are fibred by Lagrangian abelian surfaces and collect together some results about them. In Section 4 we construct a twisted Fourier–Mukai transform relating the derived category and twisted derived category of the pair of four-folds from Section 3. This is followed by our applications.

2. FM and twisted FM transforms

We begin by reviewing Mukai's work [39] on integral transforms between derived categories. We state Bridgeland's criterion [9] for when we get an equivalence of categories, and the removable singularities result of Bridgeland and Maciocia [10]. Then we review twisted Fourier–Mukai transforms, which first appeared in Căldăraru's thesis [13].

2.1. Fourier–Mukai transforms

Throughout this article, X will denote a proper and smooth scheme over $\mathbb{C}$ or a compact complex manifold; moreover, X will always be connected and will be projective unless we explicitly state otherwise. Suppose M is a proper moduli space of semi-stable sheaves on X. The moduli space is fine if there exists a universal sheaf $\mathcal{U}$ on $X \times M$

$$\begin{tikzcd} & \mathcal{U} \arrow[d] & \\ & X \times M \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_M"] & \\ X & & M. \end{tikzcd}$$

Given a sheaf $\mathcal{E}$ on M, we can define a sheaf on X by pulling $\mathcal{E}$ back to $X \times M$, tensoring with $\mathcal{U}$, and pushing down to X. This map extends to a functor, known as an *integral transform with kernel* $\mathcal{U}$, between the bounded derived categories of coherent sheaves on M and X, which we denote

$$\Phi^{\mathcal{U}}_{M\to X}: \mathcal{D}^b_{\mathrm{coh}}(M) \to \mathcal{D}^b_{\mathrm{coh}}(X)$$
$$\mathcal{E}^\bullet \mapsto \mathbf{R}\pi_{X*}\big(\mathcal{U} \overset{\mathbf{L}}{\otimes} \pi_M^*\mathcal{E}^\bullet\big).$$

Likewise, one can construct an integral transform

$$\Phi^{\mathcal{V}}_{X\to M}: \mathcal{D}^b_{\mathrm{coh}}(X) \to \mathcal{D}^b_{\mathrm{coh}}(M)$$

in the opposite direction using the object $\mathcal{V} := \mathbf{R}\mathcal{H}\mathrm{om}(\mathcal{U}, \pi_X^*\mathcal{K}_X)[\dim X]$ as kernel, provided $\mathcal{V}$ satisfies the appropriate finiteness conditions (i.e., it should have finite Tor-dimension over X and proper support over M). These functors were investigated by Mukai [39,41] in various cases: for example, when X and M are dual elliptic curves, dual abelian varieties, or K3 surfaces. In certain situations we get an equivalence of triangulated categories, and the functors are then known as *Fourier–Mukai transforms*. This phenomenon is closely related to dualities in physics, such as mirror symmetry, and may be regarded as some kind of hidden symmetry between spaces.

The following criterion, developed by Mukai, Bondal and Orlov, and Bridgeland, tells us precisely when an integral transform is a Fourier–Mukai transform. Let $\mathcal{O}_m$ denote the skyscraper sheaf supported at $m \in M$. Then $\mathcal{U}_m := \Phi^{\mathcal{U}}_{M\to X}\mathcal{O}_m$ is the sheaf on X which the point $m \in M$ represents.

Theorem 1. *(See Bridgeland [9].) Suppose that X and M are smooth projective varieties of the same dimension. The functor $\Phi^{\mathcal{U}}_{M\to X}$ is an equivalence of triangulated categories if and only if*

(1) *for all $m \in M$, $\mathcal{U}_m \otimes \mathcal{K}_X = \mathcal{U}_m$ and $\mathcal{U}_m$ is simple, i.e.,*

$$\mathrm{Hom}_X(\mathcal{U}_m, \mathcal{U}_m) = \mathbb{C},$$

(2) *and for all integers i and for all $m_1 \neq m_2 \in M$,*

$$\mathrm{Ext}^i_X(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0.$$

Remark. The conditions in the theorem are roughly akin to the requirement that $\{\mathcal{U}_m\}_{m\in M}$ behave like an orthonormal basis with respect to the $\mathrm{Ext}^\bullet$-pairing on $\mathcal{D}^b_{\mathrm{coh}}(X)$.

Example. We can take X to be a smooth elliptic curve E, and M to be its dual $\hat{E}$ (the Jacobian of E), regarded as the moduli space of degree zero line bundles on E. The Poincaré line bundle provides a universal bundle. In higher dimensions, we can take an abelian variety A, its dual $\mathrm{Pic}^0 A$, and the Poincaré line bundle. These are the original examples studied by Mukai [39].

If X is a curve or surface, techniques have been developed to determine whether a particular moduli space of sheaves M on X is smooth. In higher dimensions, however, much less is known. A priori we may have no way of knowing whether M is smooth, which is why the following result of Bridgeland and Maciocia is particularly useful. More significantly for us, it weakens slightly the conditions we need to check in order to show that an integral transform is a Fourier–Mukai transform.

Theorem 2. *(See Bridgeland and Maciocia [10], Proposition 6.1.) Suppose X is a smooth projective variety of dimension n. Let M be a fine moduli space of sheaves on X, with M an irreducible projective scheme of dimension n. Let $\mathcal{U}$ be a universal sheaf, with $\mathcal{U}_m$ defined as before. Suppose that*

(1) *for all $m \in M$, $\mathcal{U}_m \otimes \mathcal{K}_X = \mathcal{U}_m$ and $\mathcal{U}_m$ is simple,*
(2) *for all $m_1 \neq m_2 \in M$,*

$$\mathrm{Hom}_X(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0,$$

and the closed subscheme

$$\Gamma(\mathcal{U}) := \left\{(m_1, m_2) \in M \times M \;\middle|\; \mathrm{Ext}^i_X(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\right\}$$

of $M \times M$ has dimension at most $n + 1$.

Then M is smooth and

$$\Phi^{\mathcal{U}}_{M \to X} : \mathcal{D}^b_{\mathrm{coh}}(M) \to \mathcal{D}^b_{\mathrm{coh}}(X)$$

is an equivalence of categories. In particular, $\Gamma(\mathcal{U})$ must be the diagonal in $M \times M$.

2.2. Twisted Fourier–Mukai transforms

So far we have considered only fine moduli spaces. There are two kinds of obstructions to the existence of a global universal sheaf: firstly, universal sheaves may not exist locally, and secondly, the local universal sheaves may not patch together into a global sheaf. We will assume that all semi-stable sheaves are actually stable, in which case the first of these obstructions can be avoided. The second obstruction was studied by Căldăraru [13]; the main results of this subsection are quoted from his thesis.

Let M^s be the open subset of the moduli space M which parametrizes stable sheaves. Simpson proved that a universal sheaf exists locally in the étale topology on M^s (this is part (4) of Theorem 1.2.1 in [48]). This is no longer the case if we include semi-stable sheaves, as a semi-stable point in the moduli space represents a whole S-equivalence class of sheaves. In our examples, M will parametrize only stable sheaves, so $M^s = M$ and local universal sheaves will always exist on small enough open sets (in the analytic topology). So choose an open cover $\{M_i\}$ of M such that there exists a local universal sheaf $\mathcal{U}_i$ over $X \times M_i$ for all i.

Consider the restrictions of the sheaves $\mathcal{U}_i$ and $\mathcal{U}_j$ to the overlap $X \times M_{ij} := X \times (M_i \cap M_j)$

$$\begin{tikzcd}
\mathcal{U}_i \arrow[dr] & & \mathcal{U}_j \arrow[dl] \\
& X \times M_{ij} \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_M"] & \\
X & & M_{ij}.
\end{tikzcd}$$

Let $m \in M_{ij}$ and let $\mathcal{U}_m$ be the sheaf on X represented by m. Since both $\mathcal{U}_i$ and $\mathcal{U}_j$ are universal sheaves over $X \times M_{ij}$, it follows that their restrictions to $X \times m$ are isomorphic, as

$$\mathcal{U}_i|_{X\times m} \cong \mathcal{U}_m \cong \mathcal{U}_j|_{X\times m}.$$

We can assume that the open sets M_i and M_j are chosen small enough that on the intersection M_{ij} the above isomorphisms combine to give an isomorphism

$$\phi_{ij} : \mathcal{U}_i|_{X\times M_{ij}} \to \mathcal{U}_j|_{X\times M_{ij}}.$$

Since $\mathcal{U}_m$ is stable and hence simple, the isomorphism $\mathcal{U}_i|_{X\times m} \cong \mathcal{U}_j|_{X\times m}$ is unique up to multiplication by a non-zero scalar. Thus another way of formulating the above statement is to say that there is a line bundle $\mathcal{L}_{ij}$ on M_{ij} such that $\mathcal{U}_i|_{X\times M_{ij}}$ is canonically isomorphic to $\pi_M^*\mathcal{L}_{ij} \otimes \mathcal{U}_j|_{X\times M_{ij}}$. Indeed

$$\mathcal{L}_{ij} = \pi_{M*}\mathcal{H}\mathrm{om}_{X\times M_{ij}}(\mathcal{U}_j|_{X\times M_{ij}}, \mathcal{U}_i|_{X\times M_{ij}}).$$

Note that the isomorphism ϕ_{ij} itself is *not* canonical: it comes from choosing a trivialization of $\mathcal{L}_{ij}$.

On a triple intersection $X \times M_{ijk}$ composition gives

$$\phi_{ki} \circ \phi_{jk} \circ \phi_{ij} : \mathcal{U}_i|_{X\times M_{ijk}} \to \mathcal{U}_i|_{X\times M_{ijk}}.$$

On each $X \times m$ this is given by multiplication by a non-zero number, which we denote $\beta_{ijk}(m)$. Thus the composition is determined by a section $\beta_{ijk} \in \Gamma(M_{ijk}, \mathcal{O}^*)$. It can be shown that these sections give a 2-cocycle representing a cohomology class $\beta \in \mathrm{H}^2(M, \mathcal{O}^*)$.

Definition. A (holomorphic) gerbe on M is a collection of line bundles $\mathcal{L}_{ij}$ on two-fold intersections M_{ij}, for some cover $\{M_i\}$ of M, together with

(1) isomorphisms $\mathcal{L}_{ij} \cong \mathcal{L}_{ji}^{-1}$ for all i and j,
(2) and trivializations of $\mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ki}$ on M_{ijk} for all i, j, and k.

We can assume that the open sets M_i are small enough that we can choose a trivialization of each $\mathcal{L}_{ij}$ on M_{ij}. These combine to give another trivialization of $\mathcal{L}_{ij} \otimes \mathcal{L}_{jk} \otimes \mathcal{L}_{ki}$ which relative to the trivialization above is given by a section $\beta_{ijk} \in \Gamma(M_{ijk}, \mathcal{O}^*)$. Then the definition also requires β to be closed, i.e., $d\beta = 0$ where d is the Čech differential, so that it represents a cohomology class in the (cohomological) analytic Brauer group $\mathrm{H}^2(M, \mathcal{O}^*)$.

One can define isomorphism of gerbes in a natural way. Then a gerbe up to isomorphism corresponds precisely to its class in $\mathrm{H}^2(M, \mathcal{O}^*)$. Moreover, every class in $\mathrm{H}^2(M, \mathcal{O}^*)$ can be represented by a gerbe, i.e., collection of line bundles $\mathcal{L}_{ij}$ as described above. (For details see Hitchin [25], or more specifically Căldăraru [13] for the holomorphic case.)

Remark. Up to isomorphism, a line bundle corresponds to an element of $\mathrm{H}^1(M, \mathcal{O}^*)$, and hence a gerbe up to isomorphism may be regarded as a higher dimensional analogue of a line bundle.

Note that in our case

$$\mathcal{U}_i|_{X\times M_{ijk}} = \pi_M^*(\mathcal{L}_{ij}\otimes\mathcal{L}_{jk}\otimes\mathcal{L}_{ki})\otimes\mathcal{U}_i|_{X\times M_{ijk}}$$

and thus

$$\mathcal{O}_{M_{ijk}} = \mathcal{L}_{ij}\otimes\mathcal{L}_{jk}\otimes\mathcal{L}_{ki}|_{M_{ijk}}$$

meaning that $\mathcal{L}_{ij}\otimes\mathcal{L}_{jk}\otimes\mathcal{L}_{ki}|_{M_{ijk}}$ is trivial with a given trivialization. Thus our local universal sheaves lead to a gerbe $\{\mathcal{L}_{ij}\}$ on M. If we began with different local universal sheaves

$$\mathcal{U}_i' = \pi_M^*\mathcal{M}_i\otimes\mathcal{U}_i,$$

where $\mathcal{M}_i$ are line bundles on M_i, then we would have obtained different line bundles

$$\mathcal{L}_{ij}' = \mathcal{M}_i\otimes\mathcal{M}_j^{-1}\otimes\mathcal{L}_{ij}.$$

In this case $\{\mathcal{L}_{ij}\}$ and $\{\mathcal{L}_{ij}'\}$ define isomorphic gerbes (indeed, this is essentially the definition of isomorphism; it implies that the two 2-cocycles represent the same class in $\mathrm{H}^2(M,\mathcal{O}^*)$). If some choice leads to $\mathcal{L}_{ij}=\mathcal{O}_{M_{ij}}$ for all i and j, then the local universal sheaves agree on overlaps and can be patched together to give a global universal sheaf.

Proposition 3. *(See Căldăraru [13].) Let M be a moduli space of stable sheaves on X. There is a gerbe $\beta\in\mathrm{H}^2(M,\mathcal{O}^*)$ (defined up to isomorphism) representing the obstruction to the existence of a global universal sheaf on $X\times M$. This is the sole obstruction: β vanishes if and only if there exists a global universal sheaf.*

Remark. Henceforth everything we will say can be made to depend only on the isomorphism class of the gerbe, and therefore we will refer to gerbes up to isomorphism simply as gerbes. In particular, we write simply β for the cohomology class of the 2-cocycle β in $\mathrm{H}^2(M,\mathcal{O}^*)$.

Since the obstruction is completely encoded in the gerbe β, we can construct Fourier–Mukai transforms by incorporating β into the construction. Specifically, this means working with *twisted sheaves*.

Definition. Let β be a gerbe on M. A β-twisted sheaf on M is a collection of sheaves $\mathcal{F}_i$ on M_i and isomorphisms $\psi_{ij}:\mathcal{F}_i|_{M_{ij}}\to\mathcal{F}_j|_{M_{ij}}$ on the overlaps M_{ij} such that

(1) for all i and j, $\psi_{ji}=\psi_{ij}^{-1}$,
(2) and for all i, j, and k, the composition

$$\psi_{ki}\circ\psi_{jk}\circ\psi_{ij}:\mathcal{F}_i|_{M_{ijk}}\to\mathcal{F}_i|_{M_{ijk}}$$

is given by $\beta_{ijk}\,\mathrm{Id}$.

Example. The gerbe $\beta\in\mathrm{H}^2(M,\mathcal{O}^*)$ can be pulled back by π_M to give a gerbe $\pi_M^*\beta$ on $X\times M$. The collection $\{\mathcal{U}_i\}$ of local universal sheaves with isomorphisms ϕ_{ij} then gives a $\pi_M^*\beta$-twisted sheaf on $X\times M$. Let us denote this *twisted universal sheaf* simply by $\mathcal{U}$, as in the untwisted case.

The category of β-twisted sheaves over M is an abelian category, and one can construct its derived category $\mathcal{D}^b_{\mathrm{coh}}(M,\beta)$. As in the untwisted case, we can construct a functor

$$\begin{aligned}\Phi^{\mathcal{U}}_{M\to X}:\mathcal{D}^b_{\mathrm{coh}}\big(M,\beta^{-1}\big)\to \mathcal{D}^b_{\mathrm{coh}}(X)\\ \mathcal{E}^\bullet\mapsto \mathbf{R}\pi_{X*}\big(\mathcal{U}\overset{\mathbf{L}}{\otimes}\pi_M^*\mathcal{E}^\bullet\big).\end{aligned}$$

The inverse of β is defined in the obvious way

$$\big(\beta^{-1}\big)_{ijk}=(\beta_{ijk})^{-1}.$$

Note that $\mathcal{E}^\bullet$ is a complex of β^{-1}-twisted sheaves, so $\pi_M^*\mathcal{E}^\bullet$ is a complex of $\pi_M^*\beta^{-1}$-twisted sheaves; when we tensor it with $\mathcal{U}$, which is a $\pi_M^*\beta$-twisted sheaf, the twistings cancel each other, and hence

$$\mathcal{U}\overset{\mathbf{L}}{\otimes}\pi_M^*\mathcal{E}^\bullet$$

is a complex of untwisted sheaves on $X\times M$.

If $\Phi^{\mathcal{U}}_{M\to X}$ is an equivalence of triangulated categories it is called a *twisted Fourier–Mukai transform*. Căldăraru generalized Bridgeland's criterion for when this happens. First observe that the skyscraper sheaf $\mathcal{O}_m$ on M can be regarded as a twisted sheaf; simply choose the cover $\{M_i\}$ so that m lies in precisely one open set, then just one local sheaf is non-vanishing and all isomorphisms are zero. As in the untwisted case, $\mathcal{U}_m:=\Phi^{\mathcal{U}}_{M\to X}\mathcal{O}_m$ is the sheaf represented by the point $m\in M$.

Theorem 4. *(See Căldăraru [13], Theorem 3.2.1.) Suppose that X and M are smooth and of the same dimension, and $\mathcal{U}$ is a $\pi_M^*\beta$-twisted universal sheaf on $X\times M$. The functor $\Phi^{\mathcal{U}}_{M\to X}$ is an equivalence of triangulated categories if and only if*

(1) *for all $m\in M$, $\mathcal{U}_m\otimes\mathcal{K}_X=\mathcal{U}_m$ and $\mathcal{U}_m$ is simple, i.e.,*

$$\mathrm{Hom}_X(\mathcal{U}_m,\mathcal{U}_m)=\mathbb{C},$$

(2) *and for all integers i and for all $m_1\neq m_2\in M$*

$$\mathrm{Ext}^i_X(\mathcal{U}_{m_1},\mathcal{U}_{m_2})=0.$$

Remark. Căldăraru does not assume that X and M are projective. Rather, they are proper and smooth schemes over $\mathbb{C}$ or compact complex manifolds (see Definition 3.1.1 in [13]). For example, in the next subsection we will consider the case of an elliptic K3 surface X, with M the compactified relative Jacobian of X. If X does not admit any multisection then it is not projective.

2.3. Elliptic fibrations

Căldăraru [13] discusses two main examples of twisted Fourier–Mukai transforms: when X and M are K3 surfaces, and when X and M are elliptic Calabi–Yau three-folds (see also [12] and [11], respectively). We will focus on elliptic fibrations, and then generalize some of the results to fibrations by abelian varieties.

Let $p_X : X \to B$ be an elliptic fibration of arbitrary dimension. In other words, p_X is a flat fibration whose generic fibre is a smooth elliptic curve, and both X and B are smooth. Assume that all the fibres are reduced and irreducible, i.e., either they are smooth elliptic curves or they contain a single node or cusp.

Definition. The compactified relative Jacobian $J := \overline{\mathrm{Jac}}(X/B)$ of X is the moduli space parametrizing families of torsion-free rank one sheaves of degree zero on the fibres of X. This moduli space exists and is proper over the base; see D'Souza [17] or Rego [44], for instance. Note that the degree of a sheaf $\mathcal{E}$ on a (not necessarily smooth) curve C is defined by $\chi(\mathcal{E}) - \chi(\mathcal{O}_C)$; of course, for an elliptic curve X_t, $\chi(\mathcal{O}_{X_t})$ vanishes.

There is the obvious projection $p_J : J \to B$, and J is locally isomorphic to X as a fibration: it is clear that corresponding smooth fibres of X and J are isomorphic, and in fact this is true also for singular fibres (see Section 6.3 of Căldăraru [13]). Smoothness of J now follows. Note that if we allowed X to have fibres with worse singularities then J need not be smooth. The case of a type I_2 singular fibre is studied in Section 6 of Căldăraru [13]; since it is not irreducible, the elements of J supported on such a fibre need not be stable. This leads to singularities in J.

Now J can be regarded as the family of elliptic curves dual to the family $X \to B$. The fact that X and J are locally isomorphic is then a consequence of the self-duality of elliptic curves (in higher dimensions it will not always be true, as even smooth abelian varieties are not self-dual unless they are principally polarized). So we have a relative version of an elliptic curve and its dual.

Lemma 5. *Take a generic element of J and push forward by the inclusion of its support (a fibre) into X; the result is a stable sheaf on X. Let Q be the irreducible component of the Simpson moduli space of stable sheaves on X which contains this stable sheaf; if Q is not normal and reduced, replace it by the normalization of the reduced scheme associated to Q. Under the assumption that the fibres of $p_X : X \to B$ are reduced and irreducible, the compactified relative Jacobian J is isomorphic to Q.*

Remark. When the singularities that can occur in the fibres of X are fairly 'mild,' e.g., only nodes or cusps, then we expect that J will actually be a connected component of the Simpson moduli space. In other words, the connected component containing these stable sheaves should already be irreducible, normal, and reduced.

Proof. A similar example (sheaves supported on curves in a linear system on a K3 surface) was studied by Mukai: in Example 0.5 of [40] he observes that J is an open subscheme of the moduli space Q of simple (equivalent to stable in this instance) torsion sheaves on the K3 surface. Beauville [6] notes that J and Q are actually isomorphic when every curve in the linear system is reduced and irreducible. We will return to this example later in the paper. For torsion sheaves on $\mathbb{P}^2$, see Le Potier [32].

Given a torsion-free rank one sheaf of degree zero on a fibre, we can push-forward by the inclusion of the fibre into X, to obtain a torsion sheaf on X itself. According to Simpson's terminology [48], this sheaf is pure-dimensional. Since it is rank one on a fibre of X, and the fibres are reduced and irreducible, a destabilizing sheaf cannot exist. Denoting by Q the irreducible component of the Simpson moduli space which contains these stable sheaves, and by $Q^s \subset Q$ the quasi-projective subscheme of Q which parametrizes stable sheaves (we will eventually see that $Q^s = Q$), it is clear that we have an inclusion $J \hookrightarrow Q^s \subset Q$. By part (4) of Theorem 1.2.1 in Simpson [48], Q^s parametrizes a universal sheaf and satisfies a universal property, locally in the étale topology, which implies that $J \hookrightarrow Q^s \subset Q$ is actually an embedding of schemes. Since J is smooth, it will be normal and reduced; therefore the above embedding will factor through the normalization of the reduced scheme associated to Q, i.e.,

$$J \hookrightarrow \widetilde{Q}_{\mathrm{red}} \to Q_{\mathrm{red}} \to Q.$$

If Q is not already normal and reduced, we replace Q by $\widetilde{Q}_{\mathrm{red}}$ at this stage.

In order to show that J is in fact isomorphic to Q it suffices to show that J and Q have the same dimension, because both J and Q are proper and irreducible, and also normal and reduced (once we replace Q by $\widetilde{Q}_{\mathrm{red}}$, if necessary). We will show that J and Q have the same dimension by showing that the derivative of the inclusion $J \hookrightarrow Q$ is an isomorphism at a generic point. Let t be a generic point of B, let $\iota : X_t \hookrightarrow X$ be the inclusion of the smooth fibre X_t in X, and let L be a degree zero line bundle on X_t. Then L represents a point in $\mathrm{Jac}(X/B) \subset \overline{\mathrm{Jac}}(X/B) = J$, and the tangent space $T_L J$ at this point fits into the short exact sequence

$$0 \to T_L \operatorname{Jac} X_t \to T_L J \to T_t B \to 0.$$

Since X_t is smooth, $T_L(\operatorname{Jac} X_t)$ can be identified with $\mathrm{H}^1(X_t, \mathcal{O}_{X_t})$.

Next consider the tangent space to Q at $\iota_* L$. If deformations are unobstructed, then

$$T_{\iota_* L} Q \cong \operatorname{Ext}^1_X(\iota_* L, \iota_* L).$$

More generally, the tangent cone to Q at $\iota_* L$ sits inside $\operatorname{Ext}^1_X(\iota_* L, \iota_* L)$, and is cut out by equations describing the obstruction. According to Section 7.2 of Bridgeland and Maciocia [10], there is a spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_{X_t}\big(L \otimes \Lambda^q N^*_{X_t \subset X}, L\big) \Rightarrow \operatorname{Ext}^{p+q}_X(\iota_* L, \iota_* L)$$

where $N^*_{X_t \subset X}$ is the conormal bundle of the fibre X_t in X. This bundle is trivial; in fact $N^*_{X_t \subset X} = T_t^* B \otimes \mathcal{O}_{X_t}$, where $T_t^* B = N^*_{t \in B}$ is the conormal bundle of the point t in B (which is just a vector space of dimension $\dim B$). Thus the left-hand side of the spectral sequence may be written

$$\begin{aligned} E_2^{p,q} &= \operatorname{Ext}^p_{X_t}\big(L \otimes \Lambda^q\big(T_t^* B \otimes \mathcal{O}_{X_t}\big), L\big) \\ &= \operatorname{Ext}^p_{X_t}(L, L) \otimes \Lambda^q T_t B \\ &= \mathrm{H}^p(X_t, \mathcal{O}_{X_t}) \otimes \Lambda^q T_t B. \end{aligned}$$

In particular, the term $\operatorname{Ext}^1_X(\iota_* L, \iota_* L)$ fits into the long exact sequence

$$0 \to E_2^{1,0} = \mathrm{H}^1(X_t, \mathcal{O}_{X_t}) \to \operatorname{Ext}^1_X(\iota_* L, \iota_* L) \to E_2^{0,1} = T_t B \to E_2^{2,0} = 0.$$

Because of the inclusion $J \hookrightarrow Q$, we have a sequence of inclusions of tangent spaces

$$T_L J \subset T_{\iota_* L} Q \subset \operatorname{Ext}^1_X(\iota_* L, \iota_* L).$$

Comparing the exact sequences for $T_L J$ and $\mathrm{Ext}^1_X(\iota_* L, \iota_* L)$ then shows that $T_L J$ must be isomorphic to $T_{\iota_* L} Q$, which completes the proof. □

Lemma 6. *A local universal sheaf can be constructed explicitly from a local section of $X \to B$, i.e., a section over a small open set $B_i \subset B$ in the analytic topology.*

Proof. By definition, a section of $p_X : X \to B$ is a complex submanifold S of X such that $p_X|_S$ maps S isomorphically to B. Because X itself is smooth, a point q is a singular point of a singular fibre if and only if the derivative Dp_X at q does *not* surject onto $T_{p_X(q)}B$. Suppose S passes through such a point q; then $D(p_X|_S) = Dp_X|_{TS}$ at q certainly cannot surject onto $T_{p_X(q)}B$ and $p_X|_S$ cannot be an isomorphism in a neighbourhood of q. This proves that a section must intersect every fibre in its smooth locus. The same argument applies to local sections.

The absence of multiple fibres ensures that local sections always exist. Indeed, given a point q in the smooth locus of a fibre of X, we can find a local section through q, provided B_i is chosen small enough.

It is standard that a basepoint in a smooth elliptic curve allows one to normalize the Poincaré bundle, i.e., fix it up to isomorphism. This statement is also true for curves with a single node or cusp, provided the basepoint lies in the smooth locus. Our local section over B_i thus leads to a relative Poincaré line bundle (see Proposition 6.4.2 in Căldăraru [13] for details). More precisely, this is a local universal bundle on the fibre product $X \times_{B_i} J_i$, where $J_i := p_J^{-1}(B_i)$, but it can be pushed forward by the natural embedding of the fibre product into the direct product, and then we have a local universal sheaf on $X \times J_i$. □

Remark. The (twisted) Fourier–Mukai transform constructed from the relative (twisted) universal sheaf on the fibre product will be the same as that constructed from the corresponding (twisted) universal sheaf on the direct product. We usually do not need to distinguish between the two, though occasionally we will need flatness of the fibration in which case the fibre product must be employed.

As usual, these local universal sheaves need not patch together to give a global universal sheaf: the obstruction is a gerbe $\beta \in \mathrm{H}^2(J, \mathcal{O}^*)$. In this example J admits a canonical section (given by the trivial line bundle on each fibre of X) and X is a *torsor* over J, in the following sense.

Definition. In this article we use the term torsor to describe the following situation: let $p_X : X \to B$ and $p_J : J \to B$ be two fibrations, with the latter admitting a global section. Denote by X^* the open subset of X on which p_X is regular; thus X^* consists of smooth fibres and smooth points of singular fibres (J^* is defined similarly). We require that J^* be an abelian group scheme over B, with X^* a principal homogeneous space over J^*. Moreover, we require that the compactifications X and J be locally isomorphic as fibrations over B. More precisely, any local isomorphism $J_i^* \cong X_i^*$ induced by a choice of local section of $X_i^* \to B_i$, where B_i is a small open subset of B, should extend to a local isomorphism $J_i \cong X_i$.

The generic fibre could be an elliptic curve or, as we shall encounter in the next subsection, a higher dimensional abelian variety. In the case of an elliptic fibration, X is a torsor over J if the singular fibres have at worst nodes or cusps: thinking of a fibre as a plane cubic and the singularity as being a point at infinity, it is clear that an isomorphism of the affine cubics $J_t^* \cong X_t^*$ will extend to their compactifications $X_t \cong J_t$.

Note that X is isomorphic to J if and only if it admits a global section. As we saw above, the choice of a local section of X leads to a local universal sheaf (coming from a relative Poincaré line bundle); a global section then leads to a global universal sheaf. Such arguments lead to the following result.

Proposition 7. *(See Căldăraru [13].) The following are equivalent*

(1) *X is isomorphic to J,*
(2) *X admits a section,*
(3) *there is a global universal sheaf on $X \times J$,*
(4) *β vanishes.*

When β does not vanish, the fibration X can be reconstructed from J and β (this construction will be described in the next subsection). If J is projective, then X is also projective if and only if the corresponding gerbe β is a torsion element in $\mathrm{H}^2(J, \mathcal{O}^*)$, or equivalently, an element of the étale cohomology group $\mathrm{H}^2_{\text{ét}}(J, \mathcal{O}^*)$ known as the (cohomological) *Brauer group*. Thus Căldăraru's approach gives a conceptual explanation for the appearance of the Brauer group in Ogg–Shafarevich theory [15], where it is essentially the group classifying all (minimal, projective) elliptic fibrations with a given relative Jacobian J.

There are of course some subtleties: in general elliptic fibrations will have fibres which are not irreducible, nor reduced. For instance, it is not currently known whether there exists an elliptic Calabi–Yau three-fold with *only* irreducible fibres. Also, the following remark shows that not any element of the (cohomological) Brauer group can be used to construct a fibration X from J.

Remark. Gerbes on B can be pulled back to J by the projection p_J, giving an inclusion

$$\mathrm{H}^2\bigl(B, \mathcal{O}^*\bigr) \hookrightarrow \mathrm{H}^2\bigl(J, \mathcal{O}^*\bigr).$$

Moreover $J \to B$ admits a section so the inclusion splits. In Section 6 of [11] (see also Section 4.4 of [13]) Căldăraru relates the above ideas to Ogg–Shafarevich theory, which leads to the following refinement: the obstruction β really lies in $\mathrm{H}^2(J, \mathcal{O}^*)/\mathrm{H}^2(B, \mathcal{O}^*)$, or equivalently, in the kernel of the map $\mathrm{H}^2(J, \mathcal{O}^*) \to \mathrm{H}^2(B, \mathcal{O}^*)$ given by pulling back by the section $B \hookrightarrow J$.

2.4. Abelian fibrations

The main purpose of this article is to describe twisted Fourier–Mukai transforms for fibrations by abelian surfaces. We described an elliptic fibration and its relative Jacobian as a relative version of an elliptic curve and its dual. We now want to discuss the relative version of a higher dimensional abelian variety and its dual.

Let $p_X : X \to B$ be a fibration by abelian varieties, meaning that p_X is a flat fibration whose generic fibre is a smooth abelian variety, and both X and B are smooth. Once again, difficulties arise for fibres which are 'too' singular. We will assume at the very least that all fibres are reduced and irreducible (this assumption will be satisfied by our main example later on).

Definition. (See Altman and Kleiman [3].) The compactified relative Picard scheme $P := \overline{\mathrm{Pic}}^0(X/B)$ of X is the moduli space parametrizing families of torsion-free rank one sheaves of degree zero on fibres of X. We say a sheaf $\mathcal{E}$ on a fibre X_t has degree zero if it is in the same connected component of $\overline{\mathrm{Pic}}(X_t)$ as $\mathcal{O}_{X_t}$.

The following is the analogue of Lemma 5.

Lemma 8. *Under the assumption that the fibres of $p_X : X \to B$ are reduced and irreducible, the compactified relative Picard scheme P is isomorphic to an irreducible component Q of the Simpson moduli space of stable sheaves on X (where we replace Q by the normalization of the reduced scheme associated to Q, if necessary).*

Proof. The proof is almost identical to the proof of Lemma 5. One difference is that the term $E_2^{2,0} = \mathrm{H}^2(X_t, \mathcal{O}_{X_t})$ of the spectral sequence no longer vanishes. However, by comparing the two exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T_L \operatorname{Pic} X_t \arrow[r] \arrow[dd, equal] & T_L P \arrow[r] \arrow[d, phantom, "\cap"] & T_t B \arrow[r] \arrow[dd, equal] & 0 \\
& & T_{\iota_* L} Q \arrow[d, phantom, "\cap"] & & \\
0 \arrow[r] & \mathrm{H}^1(X_t, \mathcal{O}_{X_t}) \arrow[r] & \operatorname{Ext}^1_X(\iota_* L, \iota_* L) \arrow[r] & T_t B \arrow[r, "\delta"] & \mathrm{H}^2(X_t, \mathcal{O}_{X_t})
\end{tikzcd}$$

we see that we must have isomorphisms

$$T_L J \cong T_{\iota_* L} Q \cong \operatorname{Ext}^1_X(\iota_* L, \iota_* L),$$

and in particular δ must be the zero map. □

Unlike in the case of an elliptic fibration, P need not be locally isomorphic to X as a fibration, for two reasons. Firstly, if the fibres of X are not principally polarized then even a smooth fibre of P need not be isomorphic to the corresponding smooth fibre of X (they will only be isogenous). Secondly, even for principal polarizations the corresponding singular fibres of P and X may not be isomorphic.

The first problem can be resolved by taking the double dual of X, namely $X^0 := \overline{\mathrm{Pic}}^0(P/B)$. For X^0 to even be well-defined at the singular fibres, following Altman and Kleiman's definition of the compactified relative Picard scheme, we must assume that all fibres of P are reduced and irreducible. Clearly X^0 is locally isomorphic to X as a fibration away from singular fibres.

Regarding the second problem, these local isomorphisms sometimes extend over the singular fibres, and we will assume this is the case in the following discussion. This occurs when the singular fibres of X are degenerations of abelian varieties whose singularities are not too severe. In our main example the fibres of X will be Jacobians of curves with at worst nodes and/or cusps as singularities. So in fact we will be in the principally polarized case, and indeed P itself will be locally isomorphic to X as a fibration; in particular, every fibre will be self-dual. It would be interesting to find an example of a degeneration of a non-principally polarized abelian variety which is isomorphic to its double-dual. The author knows of no such example.

Remark. To construct X^0 from X in one step, we can take the relative Albanese scheme of X, following Markushevich [35]. However, use of the relative Picard scheme is essential for our moduli space interpretation.

The argument used in the proof of Lemma 6 shows that a section or local section will intersect every fibre in its smooth locus. By assumption, every fibre X_t of X is reduced and irreducible (in particular, there are no multiple fibres), and therefore X_t contains a smooth point. We can always find a local section passing through this smooth point over a sufficiently small neighbourhood of t in B, in the analytic topology. So assume that $\{B_i\}$ is an open cover of the base B, where each B_i is sufficiently small so that a local section exists over B_i. By part (iii) of Theorem 3.4 in Altman and Kleiman [2] the section can be used to fix a relative Poincaré sheaf over $X \times_{B_i} P_i$, where $P_i := p_P^{-1}(B_i)$. Let us be more specific.

The minimal compactified Picard scheme is the closure of the Picard scheme inside the compactified Picard scheme P. Part (iii) of Theorem 3.4 in [2] states that if $p_X : X \to B$ admits a global section then the minimal compactified Picard scheme parametrizes a universal sheaf; we call this a relative Poincaré sheaf since it is a Poincaré line bundle when restricted to a smooth fibre. In general, the compactified Picard scheme P could contain other irreducible components, in addition to the minimal compactified Picard scheme; however, in our case the minimal compactified Picard scheme will be the same as the compactified Picard scheme P, because we are assuming that each fibre of P is reduced and irreducible. On the other hand, in our case we only have a local section, so the relative Poincaré sheaf exists locally, over $X \times_{B_i} P_i$.

Finally, we push-forward by the embedding of $X \times_{B_i} P_i$ into $X \times P_i$ to obtain a local universal sheaf on $X \times P_i$. The obstruction to these patching together into a global universal sheaf is a gerbe $\beta \in \mathrm{H}^2(P, \mathcal{O}^*)$.

Remark. The fact that a basepoint in a smooth abelian variety allows one to normalize the Poincaré line bundle is standard (see Theorem 2.5.1 in Birkenhake and Lange [7], for example), and this can also be carried out in the relative setting (see page 598 of [7]). The result of Altman and Kleiman extends this over singular fibres. One could also appeal to part (4) of Theorem 1.2.1 in Simpson [48], which guarantees the existence of local universal sheaves for moduli spaces of stable sheaves. However, it is important to establish a correspondence between (local) sections and (local) universal sheaves, and to ensure that this correspondence applies even around singular fibres.

Proposition 9. *Let $p_X : X \to B$ be a fibration by abelian varieties all of whose fibres are reduced and irreducible. Assume that the dual fibration P also has reduced and irreducible fibres, and that the double-dual fibration X^0 is locally isomorphic to X as a fibration. More precisely, assume that X is a torsor over X^0, as defined in Section* 2.3. *The following are equivalent*

(1) *X is isomorphic to X^0,*
(2) *X admits a section,*
(3) *there is a global universal sheaf on $X \times P$,*
(4) *β vanishes.*

Proof. The equivalence of (1) and (2) is clear. To see that (2) implies (3), recall that a local section of X over B_i gives a local relative Poincaré sheaf on $X \times_{B_i} P_i$, and hence a local universal sheaf on $X \times P_i$. Then a global section will give a global relative Poincaré sheaf on $X \times_B P$ and a global universal sheaf on $X \times P$.

Proposition 3 asserts precisely the equivalence of (3) and (4). We will complete the proof by showing that (4) implies (1). First we will explain how X can be reconstructed from the gerbe β. So assume that $\beta \in \mathrm{H}^2(P, \mathcal{O}^*)$ is some gerbe arising via the above considerations. This means that there is an open cover $\{B_i\}$ of B such that β can be represented by a collection of line bundles $\mathcal{L}_{ij}$ on open sets $P_{ij} := P_i \cap P_j$, where $P_i := p_P^{-1}(B_i)$. Moreover, we can assume that the local universal sheaves $\mathcal{U}_i$ on $X \times P_i$ used to construct β come from relative Poincaré sheaves. The relation

$$\mathcal{U}_i|_{X\times P_{ij}} = \pi_P^* \mathcal{L}_{ij} \otimes \mathcal{U}_j|_{X\times P_{ij}}$$

then implies that $\mathcal{L}_{ij}$ will have degree zero when restricted to a fibre P_t of $P_{ij} \to B_{ij}$, as the Poincaré sheaves restricted to P_t are both of degree zero. Thus the restriction of $\mathcal{L}_{ij}$ to a smooth fibre P_t is a degree zero line bundle; moreover, the same is true for a singular fibre P_t (i.e., $\mathcal{L}_{ij}|_{P_t}$ will be a rank one locally free sheaf of degree zero) since $\mathcal{L}_{ij}$ itself is locally free. In either case, $\mathcal{L}_{ij}|_{P_t}$ represents a point in $\mathrm{Pic}^0(P_t)$, and thus $\mathcal{L}_{ij}$ corresponds to a local section α_{ij} of $\mathrm{Pic}^0(P/B)$ over B_{ij}. We can think of α_{ij} as a local section of $X^0 = \overline{\mathrm{Pic}}^0(P/B)$ which is contained in $\mathrm{Pic}^0(P/B) \subset \overline{\mathrm{Pic}}^0(P/B)$. In particular, this local section of X^0 intersects every fibre in its smooth locus, as it should.

Now we explain how to reconstruct X. We break X^0 up into open sets $X_i^0 := p_{X^0}^{-1}(B_i)$. Since $p_{X^0} : X^0 \to B$ has a global section, the local sections α_{ij} of $X_{ij}^0 \to B_{ij}$ can be regarded as translations in the fibres over B_{ij}. It is clear what this means in a smooth fibre. Let $X_t^0 = \overline{\mathrm{Pic}}^0(P_t)$ be a singular fibre. Recall that α_{ij} meets X_t^0 in $\mathrm{Pic}^0(P_t)$, i.e., in a point corresponding to the line bundle $\mathcal{L}_{ij}|_{P_t}$. Tensoring with this line bundle gives an automorphism of $X_t^0 = \overline{\mathrm{Pic}}^0(P_t)$. Now recall that the open subscheme X^{0*} of X^0 on which p_{X^0} is regular is an abelian group scheme (indeed X^{0*} is precisely $\mathrm{Pic}^0(P/B)$, since we assume that X^0 is locally isomorphic to X as a fibration, and therefore it has reduced and irreducible fibres). The above automorphism is translation by the section when restricted to $X_t^{0*} = \mathrm{Pic}^0(P_t)$, and thus the automorphism is an extension of this translation on X_t^{0*} to the whole of X_t^0. Finally, X is constructed by taking the collection of open sets $\{X_i^0\}$, and identifying X_i^0 and X_j^0 along the overlap X_{ij}^0 using the translations described above. Another way to describe this is that on the overlap X_i^0 is identified with X_j^0 in such a way that the zero section of X_i^0 is identified with the local section α_{ij} of X_j^0, and this makes sense because α_{ij} meets each fibre in its smooth locus.

It is clear that this procedure produces a torsor X^β over X^0. Moreover, the gerbe associated to X^β will be β, almost by definition. Since torsors over X^0 are classified by their corresponding gerbes, X^β must be X. Finally we can prove that (4) implies (1): if β vanishes then all the local sections are trivial and thus X will be isomorphic to X^0. □

Remark. As we have seen, the local sections α_{ij} must be contained in X^{0*}. In fact, they form a 1-cocycle $\alpha \in \mathrm{H}^1(B, X^{0*})$; here we have implicitly identified X^{0*} with its sheaf of local sections, which is a sheaf of abelian groups since X^{0*} is an abelian group scheme over B. By definition, each point of $X^{0*} = \mathrm{Pic}^0(P/B)$ represents a line bundle on a fibre of $P \to B$. Thus α_{ij} can be regarded as a section of $R^1 p_{P*}\mathcal{O}_P^*$ over B_{ij}, and α can be thought of as an element of

$\mathrm{H}^1(B, R^1 p_{P*}\mathcal{O}_P^*)$. It corresponds directly to $\beta \in \mathrm{H}^2(P, \mathcal{O}_P^*)$ under the Leray spectral sequence applied to the sheaf $\mathcal{O}_P^*$ and the map $p_P : P \to B$. See Section 4.4 for more details.

Remark. Another way to reconstruct X from P and β is as a moduli space of twisted sheaves. Consider the torsion sheaf $\iota_*\mathcal{O}_{P_t}$ on P, where $\iota : P_t \hookrightarrow P$ is the inclusion of a generic fibre. Since we could start with a cover $\{B_i\}$ of B such that t is contained in only one open set B_i, it follows that the gerbe β is trivializable on any fibre P_t of $P \to B$; it is only when we look at the entire family over B that the non-triviality of β becomes apparent. Therefore we can regard $\iota_*\mathcal{O}_{P_t}$ as a β-twisted sheaf on P. Moduli spaces of twisted sheaves were constructed by Lieblich [33] and Yoshioka [53]. Then X is the moduli space of β-twisted sheaves on P with (twisted) Mukai vector equal to the (twisted) Mukai vector of $\iota_*\mathcal{O}_{P_t}$. Roughly speaking, X is the β-twisted relative compactified Picard scheme of P. Since we will not use this interpretation of X, we omit the details.

3. Holomorphic symplectic manifolds

In this section we will review some examples of holomorphic symplectic manifolds. In particular, we will describe some holomorphic symplectic four-folds which are fibred by Lagrangian abelian surfaces, and collect together some facts about these spaces that will be used in the next section.

3.1. Definition and examples

Definition. Let X be a compact Kähler manifold. We call X a holomorphic symplectic manifold if it admits a closed non-degenerate two-form σ of type $(2,0)$, i.e.,

$$\sigma \in \mathrm{H}^0\big(X, \Lambda^2 T^*\big) \cong \mathrm{H}^{2,0}(X),$$

which we call a holomorphic symplectic form. Non-degeneracy means that σ induces an isomorphism between the holomorphic tangent and cotangent bundles, $T \cong T^*$. If X is simply-connected and σ generates $\mathrm{H}^{2,0}(X) \cong \mathbb{C}$ then we say that X is irreducible.

If X has (complex) dimension $2n$ then $\sigma^{\wedge n}$ trivializes the canonical bundle $\mathcal{K}_X$. By Yau's theorem, X admits a hyperkähler metric; conversely, a hyperkähler manifold is holomorphic symplectic for each choice of complex structure compatible with the hyperkähler metric. By the Bogomolov decomposition theorem [8], a holomorphic symplectic manifold has a finite cover which is the Cartesian product of a complex torus and irreducible holomorphic symplectic manifolds. In this sense, all holomorphic symplectic manifolds can be built out of irreducible ones and tori.

In dimension two, K3 surfaces are the only irreducible examples, and they form a single family up to deformation. In dimension four, there are just two currently known examples, up to deformation.

Example. The first higher dimensional example was discovered by Fujiki [20]. Let S be a K3 surface and $\mathrm{Blow}_\Delta(S \times S)$ the blow up of the diagonal. Quotienting by the involution which exchanges the two copies of S gives a smooth four-fold

$$\mathrm{Hilb}^2 S := \mathrm{Blow}_\Delta(S \times S)/\mathbb{Z}_2.$$

Fujiki showed that $\mathrm{Hilb}^2 S$ is an irreducible holomorphic symplectic four-fold. Beauville [5] generalized this example to produce an irreducible holomorphic symplectic manifold in each even dimension $2n$. These are the Hilbert schemes $\mathrm{Hilb}^n S$, which parametrize length n zero-dimensional subschemes of S, and are smooth resolutions of the symmetric products $\mathrm{Sym}^n S$.

By beginning with an abelian surface, instead of a K3 surface, Beauville [5] also constructed another family of examples, one in each even dimension, known as the generalized Kummer varieties. The following examples will also be important, though up to deformation they do not give us new spaces.

Example. Let S be a K3 surface with ample divisor H. The Mukai lattice is

$$\mathrm{H}^\bullet(S,\mathbb{Z}) = \mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathrm{H}^4(S,\mathbb{Z})$$

endowed with the bilinear form

$$\big\langle (v_0,v_2,v_4),(w_0,w_2,w_4)\big\rangle := \int_S -v_0w_4 + v_2w_2 - v_4w_0.$$

The Mukai vector of a sheaf $\mathcal{E}$, defined by $v(\mathcal{E}) := \mathrm{ch}(\mathcal{E})\mathrm{td}^{1/2} \in \mathrm{H}^\bullet(S,\mathbb{Z})$, is a convenient way to encode the topological type of the sheaf. For example, if $\mathcal{E}$ is a rank r vector bundle with Chern classes c_1 and c_2 then

$$v(\mathcal{E}) = \big(r, c_1, r + c_1^2/2 - c_2\big).$$

For fixed v in the Mukai lattice, the Mukai moduli space $\mathcal{M}^s_H(v)$ is the moduli space of stable (with respect to H) sheaves $\mathcal{E}$ on S with fixed Mukai vector $v(\mathcal{E}) = v$.

Mukai [40] showed that, for general H, $\mathcal{M}^s_H(v)$ is smooth, quasi-projective, and holomorphic symplectic of dimension $2n := \langle v,v\rangle + 2$. If v is primitive and either $v_0 > 0$ or v_2 is ample then $\mathcal{M}^s_H(v)$ is also compact. In fact it is a deformation of $\mathrm{Hilb}^n S$, and therefore it is an irreducible holomorphic symplectic manifold (these results were proved by Göttsche, Huybrechts, O'Grady, and Yoshioka; see [51] and in particular Theorem 8.1 of [52]). For non-primitive v, we can compactify to the moduli space of semi-stable sheaves $\mathcal{M}^{ss}_H(v)$, but this introduces singularities.

3.2. Lagrangian fibrations

Elliptic K3 surfaces are dense and of codimension one in the moduli space of all K3 surfaces (the locus parametrizing elliptic K3 surfaces contains countably many irreducible components). In higher dimensions we have the following result.

Theorem 10. *(See Matsushita [37,38].) Let X be a projective irreducible holomorphic symplectic manifold of dimension $2n$. Suppose $p_X : X \to B$ is a proper surjective morphism, whose generic fibre is connected, and with smooth projective base B of dimension strictly between 0 and $2n$. Then*

(1) *the generic fibre is a (holomorphic) Lagrangian abelian variety of dimension n,*
(2) *the base is Fano with the same Hodge numbers as $\mathbb{P}^n$.*

In particular, when $n = 2$ the base is $\mathbb{P}^2$.

We call $p_X : X \to B$ a *Lagrangian fibration*. It is currently an open problem whether an arbitrary holomorphic symplectic manifold can be deformed to a Lagrangian fibration. This is possible for all the known irreducible holomorphic symplectic manifolds. In particular, for $\mathrm{Hilb}^n S$ we have the following example, known as the Beauville–Mukai integrable system.

Example. Let S be a K3 surface which contains a smooth genus $g \geqslant 2$ curve C which is also an ample divisor on S. Then C moves in a g-dimensional linear system $|C| \cong \mathbb{P}^g$, and taking $H = C$ gives us an embedding $S \hookrightarrow (\mathbb{P}^g)^\vee$ (unless $g = 2$, in which case we instead get a double cover of the plane). Let Z be the Mukai moduli space $\mathcal{M}^s_H((0, [C], 1))$, where $[C]$ denotes the class of C in $\mathrm{H}^2(S, \mathbb{Z})$. Then Z is smooth and compact. The typical element is the push-forward of a degree g line bundle on a smooth curve $D \in |C|$. Thus Z is fibred over $|C| \cong \mathbb{P}^g$, and the generic fibre is a smooth abelian variety of dimension g, namely the degree g Picard group $\mathrm{Pic}^g D$ of a smooth genus g curve

$$\begin{tikzcd} \mathrm{Pic}^g \arrow[r, hook] & Z \arrow[d] \\ & |C| \cong \mathbb{P}^g. \end{tikzcd}$$

We claim that Z is birational to $\mathrm{Hilb}^g S$. A generic element of Z gives a generic degree g line bundle on a smooth genus g curve $D \subset S$. This line bundle will have a unique section, up to scale, which will vanish at precisely g distinct points. This gives us a rational map $Z \dashrightarrow \mathrm{Hilb}^g S$.

In the other direction, a generic element of $\mathrm{Hilb}^g S$ consisting of g distinct points determines a hyperplane in $(\mathbb{P}^g)^\vee$. For $g > 2$ this hyperplane cuts $S \subset (\mathbb{P}^g)^\vee$ in a smooth curve of the linear system $|C|$; for $g = 2$ it can be pulled back from $(\mathbb{P}^2)^\vee$ to give such a curve. Moreover, the g points lie on the curve and determine a degree g line bundle on the curve. Thus we obtain a rational map $\mathrm{Hilb}^g S \dashrightarrow Z$, establishing the claim.

This example was used by Beauville [6] to count the number of rational curves (with nodes) in each linear system $|C|$. If we assume that $\mathrm{Pic}\, S \cong \mathbb{Z}C$ then every curve in the linear system $|C|$ will be reduced and irreducible. In this case Z can be identified with the compactified relative Jacobian $\overline{\mathrm{Pic}}^g(\mathcal{C}/|C|)$ (see D'Souza [17] or Altman, Iarrobino, and Kleiman [1]) of the family of curves $\mathcal{C} \to |C|$ in the linear system; the proof is the same as the proof of Lemma 5. Huybrechts [26] showed that birational holomorphic symplectic manifolds have the same periods and represent non-separated points in their moduli space of deformations, which is non-Hausdorff. It follows that Z and $\mathrm{Hilb}^g S$ are also deformation equivalent.

Debarre [14] used similar methods as Beauville to count the number of genus two curves (with nodes) in a linear system on an abelian surface. This included showing that the generalized Kummer varieties can be deformed to Lagrangian fibrations (see also Example 3.8 in [45]).

3.3. More about Z

In this subsection we concentrate on the fibrations by Lagrangian abelian surfaces which are deformations of $\mathrm{Hilb}^2(S)$. We collect together some facts that will be of use in the next section.

Let $S \to (\mathbb{P}^2)^\vee$ be a hyperelliptic K3, i.e., a double cover of the plane ramified over a sextic δ. We will assume that δ is generic; in particular it does not admit a tritangent (this is a codimension one condition on the space of plane sextics). The pull-back of a generic line in $(\mathbb{P}^2)^\vee$ gives a smooth genus two curve C in S, whose linear system is the $\mathbb{P}^2$ dual to $(\mathbb{P}^2)^\vee$. Genericity of δ implies that $\operatorname{Pic} S \cong \mathbb{Z}C$, and we take $H = C$ as an ample divisor on S. Let us use Z^2 to denote the moduli space $Z = \mathcal{M}^s_H((0, [C], 1))$ described in the previous subsection. Thus Z^2 is a four-fold fibred by abelian surfaces

$$\begin{tikzcd} \mathrm{Pic}^2 \arrow[r, hook] & Z^2 \arrow[d] \\ & {|C| \cong \mathbb{P}^2.} \end{tikzcd}$$

For all $d \in \mathbb{Z}$, we can construct a similar fibration Z^d, whose generic fibre is the degree d Picard group $\operatorname{Pic}^d D$ of a smooth genus two curve $D \in |C|$

$$\begin{tikzcd} \mathrm{Pic}^d \arrow[r, hook] & Z^d \arrow[d] \\ & {|C| \cong \mathbb{P}^2.} \end{tikzcd}$$

Indeed, Z^d is just the Mukai moduli space $\mathcal{M}^s_H((0, [C], d-1))$. Since $[C]$ is primitive in $\mathrm{H}^2(S, \mathbb{Z})$, the Mukai vector $(0, [C], d-1)$ must be primitive, and Z^d is a smooth compact irreducible holomorphic symplectic four-fold. Moreover, Z^d is deformation equivalent to $\operatorname{Hilb}^2(S)$ (see Theorem 8.1 of Yoshioka [52], for example).

The fact that $\operatorname{Pic} S \cong \mathbb{Z}C$ implies that every curve in the family $\mathcal{C} \to |C|$ is reduced and irreducible. As in Lemma 5, Z^d can be identified with the compactified relative Jacobian $\overline{\operatorname{Pic}}^d(\mathcal{C}/|C|)$, and Markushevich [34,35] gave explicit constructions of Z^0 and Z^1 via this approach (we will see shortly that the other spaces Z^d are each isomorphic to one of these two). Note that the curves in $|C|$ are

(1) smooth genus two curves, generically; pull-backs of lines in $(\mathbb{P}^2)^\vee$ meeting δ transversely,
(2) genus one curves with one node, in codimension one; pull-backs of lines tangent to δ at precisely one point,
(3) genus one curves with one cusp, in codimension two; pull-backs of flex lines of δ,
(4) and rational curves with two nodes, in codimension two; pull-backs of bitangents to δ.

The singular curves sit above the curve $\Delta \subset \mathbb{P}^2$ dual to δ. By the Plücker formulae [21] Δ is a degree 30 curve with 72 cusps and 324 nodes. The type (3) and (4) singular curves sit above the cusps and nodes respectively.

As remarked above, for generic δ every curve is geometrically integral (i.e., reduced and irreducible); this can be observed directly for curves of type (1) to (4) above. Altman, Iarrobino, and Kleiman [1] proved that for families of geometrically integral curves embedded in surfaces, the fibres of $\overline{\operatorname{Pic}}^d(\mathcal{C}/|C|)$ are always irreducible. In fact, this follows from the weaker result of D'Souza [17] since our curves have at worst nodes or cusps as singularities; D'Souza also showed that the fibres are reduced and equidimensional in this case. In fact we will give an explicit description of all the fibres.

Clearly for a smooth genus two curve D, of type (1), the Jacobian $\mathrm{Pic}^d D$ is already compact, and is a smooth abelian surface. For type (2) we have the following description.

Lemma 11. *(See Oda and Seshadri [42], Example (1) on p. 83.) Let D be a genus one curve with one node r, i.e., of arithmetic genus two. Let $\pi : \tilde{D} \to D$ be the normalization of D, and let p and q be the two points of $\tilde{D}$ which are identified by π. Let $\mathcal{L}$ be the Poincaré line bundle on $\tilde{D} \times \mathrm{Pic}^0 \tilde{D}$, and let $\mathcal{L}_p$ and $\mathcal{L}_q$ be the restrictions to $\{p\} \times \mathrm{Pic}^0 \tilde{D}$ and $\{q\} \times \mathrm{Pic}^0 \tilde{D}$ respectively. Then $\overline{\mathrm{Pic}}^0 D$ is given by taking the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{L}_p \oplus \mathcal{L}_q)$ over $\mathrm{Pic}^0 \tilde{D}$ and identifying $s_0 := \mathbb{P}(\mathcal{L}_p) \cong \mathrm{Pic}^0 \tilde{D}$ and $s_\infty := \mathbb{P}(\mathcal{L}_q) \cong \mathrm{Pic}^0 \tilde{D}$ with a translation by $\mathcal{O}(p-q)$. (Note that $\mathcal{O}(p-q)$ is a point on $\mathrm{Pic}^0 \tilde{D}$, by which we can translate.)*

Next consider type (3) curves.

Lemma 12. *(See Kleiman [28], Theorem 10.) Let D be a genus one curve with one cusp r, i.e., of arithmetic genus two. Let $\pi : \tilde{D} \to D$ be the normalization of D, and let $p \in \tilde{D}$ be the preimage of r. Let $\mathcal{L}$ be the Poincaré line bundle on $\tilde{D} \times \mathrm{Pic}^0 \tilde{D}$ and $\mathcal{L}_p$ the restrictions to $\{p\} \times \mathrm{Pic}^0 \tilde{D}$. Let $J^1\mathcal{L}_p$ denote the first jet bundle of $\mathcal{L}_p$; there is an exact sequence*

$$0 \to \Omega^1_{\mathrm{Pic}^0 \tilde{D}} \otimes \mathcal{L}_p \to J^1\mathcal{L}_p \to \mathcal{L}_p \to 0.$$

The $\mathbb{P}^1$-bundle $\mathbb{P}(J^1\mathcal{L}_p)$ over $\mathrm{Pic}^0 \tilde{D}$ has a section $s_\infty := \mathbb{P}(\Omega^1 \otimes \mathcal{L}_p) \cong \mathrm{Pic}^0 \tilde{D}$. Then $\overline{\mathrm{Pic}}^0 D$ is given by identifying s_∞ with an infinitesimal translation of itself, thereby producing a locus of cusps. More precisely, $\mathbb{P}(J^1\mathcal{L}_p)$ is the normalization of $\overline{\mathrm{Pic}}^0 D$, they are isomorphic away from s_∞, and the kernel of the differential of the normalization map is a certain vector field along s_∞ (i.e., a section of the tangent bundle of $\mathbb{P}(J^1\mathcal{L}_p)$ restricted to s_∞) which is described in Kleiman's paper [28] *(see also Altman and Kleiman* [4]*).*

Finally, consider type (4) curves.

Lemma 13. *(See Oda and Seshadri [42], Example (2) on p. 83.) Let D be a rational curve with two nodes r_1 and r_2, i.e., of arithmetic genus two. Let $\pi : \tilde{D} \to D$ be the normalization of D, and let $\{p_1, q_1\}$ and $\{p_2, q_2\}$ be the pairs of points of $\tilde{D}$ which are identified by π. Since $\tilde{D} \cong \mathbb{P}^1$, we can define λ to be the cross-ratio of the four points $\{p_1, q_1, p_2, q_2\}$. Note that multiplication by λ gives an isomorphism $\mathbb{P}^1 \to \mathbb{P}^1$ which fixes 0 and ∞. Then $\overline{\mathrm{Pic}}^0 D$ is given by taking $\mathbb{P}^1 \times \mathbb{P}^1$ and identifying $s^1_0 := \{0\} \times \mathbb{P}^1$ and $s^1_\infty := \{\infty\} \times \mathbb{P}^1$ via multiplication by λ, and identifying $s^2_0 := \mathbb{P}^1 \times \{0\}$ and $s^2_\infty := \mathbb{P}^1 \times \{\infty\}$ via multiplication by λ.*

Remark. Choose a point p_0 in the smooth locus of D. Then $\mathcal{O}_D(dp_0)$ is a degree d line bundle on D. If we tensor a torsion-free rank one sheaf of degree zero with $\mathcal{O}_D(dp_0)$ then we get a torsion-free rank one sheaf of degree d. In fact this gives an isomorphism

$$\overline{\mathrm{Pic}}^0 D \cong \overline{\mathrm{Pic}}^d D$$

(which is not canonical since it depends on the choice of p_0). So the above lemmas serve to describe the compactified Jacobians of all degrees.

Remark. Only the compactified Jacobian of type (4) has non-zero Euler characteristic, equal to one. Therefore only these fibres make a non-trivial contribution to the Euler characteristic of Z^d, which is therefore 324 (see Beauville [6] for how this method can be used to calculate the Euler characteristic of $\mathrm{Hilb}^n S$).

Remark. A fact that we will use later is that $Z^d \to \mathbb{P}^2$ is a flat fibration. This follows from the corollary after Theorem 23.1 in Matsumura's book [36], since Z^d is smooth and the fibres are equidimensional.

The first of the above remarks leads to the following lemma.

Lemma 14. *All of the spaces Z^{2m} (for $m \in \mathbb{Z}$) are isomorphic and all of the spaces Z^{2m+1} (for $m \in \mathbb{Z}$) are isomorphic, so we essentially have just two spaces Z^0 and Z^1. Moreover, Z^1 is a torsor over Z^0.*

Proof. First note that Z^0 admits a global section, given by taking the trivial (degree zero) line bundle $\mathcal{O}_D$ on each curve $D \in |C|$. Choose an open set $U \subset |C|$ in the analytic topology over which there is a local section of $\mathcal{C} \to |C|$; as usual, the local section intersects each curve in its smooth locus. The local section then gives a local isomorphism $Z^0/U \to Z^d/U$, and hence all the spaces Z^d are locally isomorphic to Z^0 as fibrations. In particular, Z^d is a torsor over Z^0.

Now for all $d \in \mathbb{Z}$ there is a global isomorphism

$$Z^d \xrightarrow{\cong} Z^{d+2}$$

which over $D \in |C|$ is given by tensoring stable sheaves with the canonical bundle $\mathcal{K}_D$ (of degree two). This completes the proof. □

Proposition 15. *For a generic hyperelliptic K3 surface S (i.e., the sextic δ is generic), the spaces Z^0 and Z^1 are not isomorphic. Indeed they have different periods, and hence are not even birational.*

Proof. We will use O'Grady's description [43] of the weight two Hodge structure of the Mukai moduli space $\mathcal{M}^s_H(v)$ to show that Z^0 and Z^1 have non-isomorphic Picard lattices, and hence different periods. The Mukai lattice $\mathrm{H}^\bullet(S,\mathbb{Z})$ can be given the Hodge structure whose $(2,0)$, $(1,1)$, and $(0,2)$ components are

$$\mathrm{H}^{2,0}(S), \quad \mathrm{H}^{0,0}(S) \oplus \mathrm{H}^{1,1}(S) \oplus \mathrm{H}^{2,2}(S), \quad \text{and} \quad \mathrm{H}^{0,2}(S)$$

respectively. O'Grady proved that for $\langle v, v\rangle > 0$, the weight two Hodge structure of $\mathcal{M}^s_H(v)$ is isomorphic to $v^\perp$, and the induced quadratic form on $v^\perp$ agrees with the Beauville–Bogomolov quadratic form on $\mathrm{H}^2(\mathcal{M}^s_H(v),\mathbb{Z})$. For generic S the Picard lattice $\mathrm{H}^{1,1}(S) \cap \mathrm{H}^2(S,\mathbb{Z})$ is generated by $[C]$. Then the Picard lattice of $\mathcal{M}^s_H(v)$ is isomorphic to

$$\bigl\{(a,b,c) \in \mathbb{Z}^3 \bigm| \bigl(a, b[C], c\bigr) \in v^\perp\bigr\}.$$

For the particular spaces we are interested in, we find the Picard lattices of $Z^0 = \mathcal{M}^s_H((0,[C],-1))$ and $Z^1 = \mathcal{M}^s_H((0,[C],0))$ are

$$\mathbb{Z}\bigl(-2,[C],0\bigr) \oplus \mathbb{Z}(0,0,1)$$

and

$$\mathbb{Z}(-1,0,0) \oplus \mathbb{Z}(0,0,1)$$

respectively, and the induced quadratic forms

$$\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are easily seen to be non-isomorphic; for example, only the second is unimodular. □

Corollary 16. *Generically, Z^1 does not admit a section. It does, however, admit a rational* 2-*valued section.*

Proof. If Z^1 admitted a section it would be isomorphic to Z^0. A 2-valued section is a pair of degree one line bundles on each curve $D \in |C|$. Note that a degree two line bundle (such as $\mathcal{K}_D$) on each curve does *not* canonically give a pair of degree one line bundles. However, by adjunction

$$\mathcal{K}_D = \mathcal{O}(D)|_D = \mathcal{O}(C)|_D$$

as $\mathcal{K}_S$ is trivial. In other words, as a divisor $C|_D$ is just a pair of points on each curve $D \neq C$, giving us a pair of degree one line bundles as required. □

Remark. This is only a rational 2-valued section as obviously C does not intersect itself transversely. A genuine 2-valued section appears not to exist, though this will not present any difficulties. From the discussion in Section 2, we know that there should exist an element $\alpha \in \mathrm{H}^1(\mathbb{P}^2, Z^{0*})$ classifying the torsor Z^1. The fact that Z^1 admits a rational 2-valued section means that α should be 2-torsion, at least in étale cohomology $\mathrm{H}^1_{\text{ét}}(\mathbb{P}^2, Z^{0*})$. This α is precisely the element of $\mathrm{H}^1_{\text{ét}}(\mathbb{P}^2, Z^{0*})$ constructed explicitly by Markushevich in Theorem 5.1 of [35] (it is called σ in his paper).

Remark. An example of a non-generic hyperelliptic K3 surface S arises when the sextic admits a tritangent: this condition is codimension one on the space of sextics. In this case the pull-back of the tritangent to S gives a reducible curve, consisting of two rational curves C_1 and C_2 which intersect transversely at three points. Let us assume that S is otherwise as generic as possible, so that its Picard lattice is generated by $[C_1]$ and $[C_2]$ (since $[C] = [C_1] + [C_2]$ and $C^2 = 2$, $[C_1]$ and $[C_2]$ cannot be proportional). The Picard lattice of $\mathcal{M}^s_H(v)$ is therefore isomorphic to

$$\bigl\{(a,b,c,d) \in \mathbb{Z}^4 \bigm| \bigl(a, b[C_1] + c[C_2], d\bigr) \in v^{\perp}\bigr\}.$$

A bit of work shows that now Z^0 and Z^1 have isomorphic Picard lattices, with quadratic form

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -10 \end{pmatrix}$$

with respect to the bases

$$\mathbb{Z}\bigl(-1, [C_1], 1\bigr) \oplus \mathbb{Z}\bigl(-5, 4[C_1] + [C_2], 1\bigr) \oplus \mathbb{Z}\bigl(-10, 9[C_1] + [C_2], 5\bigr)$$

and

$$\mathbb{Z}(-1, 0, 0) \oplus \mathbb{Z}(0, 0, 1) \oplus \mathbb{Z}\bigl(0, [C_1] - [C_2], 0\bigr)$$

respectively.

In this case, there is a birational map $Z^0 \dashrightarrow Z^1$ inducing the isomorphism of periods: since $C_1.C = 1$ the restriction of the line bundle $\mathcal{O}(C_1)$ on S to a generic curve $D \in |C|$ is a degree one line bundle, and tensoring with this line bundle induces the birational map. This line bundle also provides a (rational) section of Z^1.

4. Derived equivalences

In Section 3 we described a certain four-fold Z^0 which is fibred by abelian surfaces, as well as another fibration Z^1 which is a torsor over Z^0. In this section we will reinterpret this torsor via moduli spaces and twisted Fourier–Mukai transforms. In particular, we will show that the derived category of sheaves on Z^1 is equivalent to the derived category of twisted sheaves on Z^0. We will also show that Z^1 can be deformed to Z^0 via Lagrangian fibrations.

4.1. The dual fibration

Since Z^1 is fibred over $\mathbb{P}^2$, we can define its compactified relative Picard scheme, which we regard as a dual fibration. Our first goal is to understand this space. Firstly, observe that Z^0 and Z^1 are locally isomorphic as fibrations, and therefore their dual fibrations $\overline{\mathrm{Pic}}^0(Z^0/\mathbb{P}^2)$ and $\overline{\mathrm{Pic}}^0(Z^1/\mathbb{P}^2)$ must also be locally isomorphic. Since these dual fibrations both admit canonical global sections (given by the trivial bundle on each fibre of Z^0, respectively, Z^1), we must have

$$\overline{\mathrm{Pic}}^0\bigl(Z^0/\mathbb{P}^2\bigr) \cong \overline{\mathrm{Pic}}^0\bigl(Z^1/\mathbb{P}^2\bigr).$$

In other words, we get the same dual fibration P, regardless of whether we begin with Z^0 or Z^1. In this section we will show that P is isomorphic to Z^0; thus Z^0 is self-dual. By the above observations, this also implies that Z^0 is the dual fibration of Z^1 (there is no contradiction here: the operation of taking the dual fibration is only locally reflexive). Thus we will interpret Z^0, which is a priori a moduli space of sheaves on the K3 surface S, as also being a moduli space of stable sheaves on the four-fold Z^1.

It is clear that corresponding smooth fibres of Z^0 and P are isomorphic: if $\operatorname{Pic}^0 D$ is a smooth fibre of Z^0 (i.e., D is of type (1), a smooth genus two curve), then the corresponding fibre of P is

$$\operatorname{Pic}^0\big(\operatorname{Pic}^0 D\big) \cong \operatorname{Pic}^0 D.$$

This result also extends to the singular fibres by the following autoduality result.

Theorem 17. *(See Esteves and Kleiman [18].) Let D be a (reduced and irreducible) surficial curve with at worst nodes and/or cusps as singularities. Then the compactified Picard scheme of the compactified Jacobian of D is isomorphic to the compactified Jacobian of D, i.e.,*

$$\overline{\operatorname{Pic}}^0\big(\overline{\operatorname{Pic}}^0 D\big) \cong \overline{\operatorname{Pic}}^0 D.$$

Remark. The compactified Jacobian means $\overline{\operatorname{Pic}}^0$ of a curve, whereas the compactified Picard scheme means $\overline{\operatorname{Pic}}^0$ of a higher dimensional scheme. Thus the wording of the theorem implicitly assumes that the genus of D is at least two, so that $\overline{\operatorname{Pic}}^0 D$ has dimension at least two, though the theorem is also true for elliptic curves.

Remark. In an earlier paper Esteves, Gagné, and Kleiman [19] proved that if D is a surficial curve with at worst double points as singularities (which includes nodes or cusps), then

$$\operatorname{Pic}^0\big(\overline{\operatorname{Pic}}^0 D\big) \cong \operatorname{Pic}^0 D.$$

Now it is not immediately true that the same result must hold for compactified Picard schemes. A priori, a compactified Picard scheme could consist of several irreducible components, with whole components parametrizing torsion-free rank one sheaves *none* of which are locally free. Such examples even exist (see Altman, Iarrobino, and Kleiman [1] for an example of a space curve whose compactified Jacobian has this property), but only in the presence of particularly bad kinds of singularities. It was also shown in [1] that the compactified Jacobian of a surficial curve must be irreducible. So when the singularities are mild, as in the theorem, one expects much better behaviour.

In particular, Theorem 17 applies to our curves of type (2)–(4).

Corollary 18. *The spaces P and Z^0 are isomorphic. In particular, P is a smooth holomorphic symplectic four-fold.*

Proof. Because of Theorem 17, we know that

$$\overline{\operatorname{Pic}}^0\big(\overline{\operatorname{Pic}}^0 D\big) \cong \overline{\operatorname{Pic}}^0 D$$

for all four kind of curves. Moreover, Esteves and Kleiman's result (Theorem 4.1 in [18]) also holds in the relative case. So for the family of curves $\mathcal{C} \to \mathbb{P}^2$, we have

$$\overline{\operatorname{Pic}}^0\big(\overline{\operatorname{Pic}}^0(\mathcal{C}/\mathbb{P}^2)\big) \cong \overline{\operatorname{Pic}}^0\big(\mathcal{C}/\mathbb{P}^2\big)$$

which is precisely $P \cong Z^0$. $\square$

4.2. A Fourier–Mukai transform

We have seen that P parametrizes stable sheaves on Z^0. Moreover, since Z^0 admits a global section, there exists a universal sheaf $\mathcal{U}$ on $Z^0 \times P$. We can therefore construct the integral transform with kernel $\mathcal{U}$

$$\Phi^{\mathcal{U}}_{P\to Z^0} : \mathcal{D}^b_{\mathrm{coh}}(P) \to \mathcal{D}^b_{\mathrm{coh}}\bigl(Z^0\bigr).$$

Theorem 19. *The functor $\Phi^{\mathcal{U}}_{P\to Z^0}$ is an equivalence of triangulated categories.*

Proof. We will apply Bridgeland and Maciocia's Theorem 2. Let $\mathcal{U}_m := \Phi^{\mathcal{U}}_{P\to Z^0}\mathcal{O}_m$ be the sheaf on Z^0 which the point $m \in P$ represents. We must show

(1) for all $m \in P$, $\mathcal{U}_m \otimes \mathcal{K}_{Z^0} = \mathcal{U}_m$ and $\mathcal{U}_m$ is simple,
(2) for all $m_1 \neq m_2 \in P$,

$$\mathrm{Hom}_{Z^0}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0,$$

and the closed subscheme

$$\Gamma(\mathcal{U}) := \bigl\{(m_1, m_2) \in P \times P \bigm| \mathrm{Ext}^i_{Z^0}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) \neq 0 \text{ for some } i \in \mathbb{Z}\bigr\}$$

of $P \times P$ has dimension at most five.

Since $\mathcal{U}_m$ is stable, it is also simple. Since Z^0 is holomorphic symplectic, it has trivial canonical bundle $\mathcal{K}_{Z^0}$. Thus condition (1) follows.

If $\mathcal{U}_{m_1} \to \mathcal{U}_{m_2}$ is a non-trivial morphism, then it must be an isomorphism since $\mathcal{U}_{m_2}$ is stable. Therefore $m_1 = m_2$, proving the first part of condition (2). It remains to prove the bound on the dimension of $\Gamma(\mathcal{U})$.

Firstly, suppose that m_1 and m_2 lie in different fibres of P. Then $\mathcal{U}_{m_1}$ and $\mathcal{U}_{m_2}$ are sheaves supported on different (disjoint) fibres of Z^0. Therefore

$$\mathrm{Ext}^i_{Z^0}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0$$

for all i because all local $\mathcal{E}xt$ sheaves vanish.

Next suppose that m_1 and m_2 lie in the same *smooth* fibre of P. Then $\mathcal{U}_{m_1}$ and $\mathcal{U}_{m_2}$ are supported on the smooth fibre $Z^0_t = \mathrm{Pic}^0 D$ of Z^0; in fact they are of the form $\iota_* L_1$ and $\iota_* L_2$ respectively, where $\iota : Z^0_t \hookrightarrow Z^0$ is inclusion and L_1 and L_2 are degree zero line bundles on the (smooth) abelian surface Z^0_t. Now we have the following spectral sequence, taken from Section 7.2 of Bridgeland and Maciocia [10]

$$E_2^{p,q} := \mathrm{Ext}^p_{Z^0_t}\bigl(L_1 \otimes \Lambda^q \mathcal{O}^{\oplus 2}_{Z^0_t}, L_2\bigr) \Rightarrow \mathrm{Ext}^{p+q}_{Z^0}(\iota_* L_1, \iota_* L_2)$$

where $\mathcal{O}_{Z_t^0}^{\oplus 2}$ is really the conormal bundle of Z_t^0 in Z^0, which is trivial. If $m_1 \neq m_2$ then L_1 and L_2 are not isomorphic. The cohomology of the non-trivial degree zero line bundle $L_1^\vee \otimes L_2$ on Z_t^0 therefore vanishes in all degrees (see Chapter 3 of Birkenhake and Lange [7]), i.e.,

$$\mathrm{H}^p\big(Z_t^0, L_1^\vee \otimes L_2\big) = 0 \quad \text{for all } p \in \mathbb{Z}.$$

It follows that all terms $E_2^{p,q}$ of the spectral sequence vanish and we have proved

$$\mathrm{Ext}^i_{Z^0}(\mathcal{U}_1, \mathcal{U}_2) = \mathrm{Ext}^i_{Z^0}(\iota_* L_1, \iota_* L_2) = 0$$

for all $i \in \mathbb{Z}$ in this case.

We have shown that $\Gamma(\mathcal{U})$ is a subset of

$$\mathrm{Diag} \cup \big\{(m_1, m_2) \in P \times P \;\big|\; m_1 \text{ and } m_2 \text{ lie in the same singular fibre}\big\}$$

where Diag is the diagonal in $P \times P$. But the singular fibres of P sit above the curve $\Delta \subset \mathbb{P}^2$ and have dimension two. Thus

$$\big\{(m_1, m_2) \in P \times P \;\big|\; m_1 \text{ and } m_2 \text{ lie in the same singular fibre}\big\}$$

has dimension five, and Diag clearly has dimension four. Therefore $\Gamma(\mathcal{U})$ has dimension at most five and the theorem is proved. □

Remark. Since $P \cong Z^0$ by Corollary 18, the equivalence

$$\Phi^{\mathcal{U}}_{P \to Z^0} : \mathcal{D}^b_{\mathrm{coh}}(P) \to \mathcal{D}^b_{\mathrm{coh}}\big(Z^0\big)$$

is really an auto-equivalence of the derived category of Z^0. It is non-trivial: for instance, on a smooth fibre it induces a non-trivial auto-equivalence of the derived category of a principally polarized abelian surface (which was first constructed by Mukai [39]).

In fact, we can extend Mukai's results to degenerate abelian surfaces as follows. Let D be an arbitrary curve of type (2), (3), or (4). We claim that D occurs in some hyperelliptic K3 surface S.

Lemma 20. *Let D be a genus two curve, either smooth or singular of type* (2), (3)*, or* (4)*. Then D is contained in a hyperelliptic K3 surface S which is the double cover of* $(\mathbb{P}^2)^\vee$ *branched over a smooth sextic δ. Moreover, we can assume that the sextic δ does not admit a tritangent, so that the K3 surface S is still sufficiently generic.*

Proof. Write D as a double cover of $\mathbb{P}^1$ branched over six points $\lambda_1, \dots, \lambda_6 \in \mathbb{P}^1$. Some of these points may coincide (a pair of points will coincide for D of type (2), three points will coincide for type (3), and two pairs will coincide for type (4)), but we can assume that none of the points is at infinity in $\mathbb{P}^1$. Next we embed $\mathbb{P}^1$ in $(\mathbb{P}^2)^\vee$ as the line $z = 0$ in homogeneous coordinates $[x, y, z]$. A sextic δ which intersects this line at $\lambda_1, \dots, \lambda_6$ will have equation

$$f(x, y, z) = (x - \lambda_1 y) \cdots (x - \lambda_6 y) + z g(x, y, z)$$

where $g(x, y, z)$ is a degree five homogeneous polynomial. If we choose $g(x, y, z)$ generically the sextic will be smooth, and we can let S be the double cover of $(\mathbb{P}^2)^\vee$ branched over δ. It remains to show that δ does not admit a tritangent.

A smooth sextic will admit a finite number of bitangents: they correspond to nodes of the dual curve. Some of these bitangents may be tritangents, i.e., they may be tangent to the sextic at a third point. Suppose that for some choice of $g(x, y, z)$ above the corresponding sextic admits a tritangent. Without loss of generality, we can assume that the tritangent is the line $y = 0$, with δ meeting it at the points $x/z = \mu_1, \mu_2, \mu_3$, each with multiplicity two. Therefore we can write

$$f(x, 0, z) = x^6 + zg(x, 0, z) = (x - \mu_1 z)^2(x - \mu_2 z)^2(x - \mu_3 z)^2.$$

If we now deform $g(x, y, z)$ by adding

$$-t^2 z(x - \mu_1 z)^2(x - \mu_2 z)^2$$

where t is the deformation parameter, we get

$$\begin{aligned} f_t(x, 0, z) &= (x - \mu_1 z)^2(x - \mu_2 z)^2(x - \mu_3 z)^2 - t^2 z^2(x - \mu_1 z)^2(x - \mu_2 z)^2 \\ &= (x - \mu_1 z)^2(x - \mu_2 z)^2\big(x - (\mu_3 + t)z\big)\big(x - (\mu_3 - t)z\big). \end{aligned}$$

So for small $t \neq 0$ the line $y = 0$ is still tangent to the sextic at μ_1 and μ_2, but it is no longer tangent to the sextic at a third point. This shows that a small deformation destroys the tritangent, and therefore the generic sextic δ will not admit a tritangent. □

Using the above K3 surface S, we construct Z^0 and P as before. The compactified Jacobian $J := \overline{\mathrm{Pic}}^0 D$ occurs as a fibre of Z^0, and its dual $\overline{\mathrm{Pic}}^0 J$ is the corresponding fibre of P. We will use ι to denote the inclusion of $J \times \overline{\mathrm{Pic}}^0 J$ into $Z^0 \times P$. The universal sheaf $\mathcal{U}$ on $Z^0 \times P$ restricts to a universal sheaf $\iota^*\mathcal{U}$ on $J \times \overline{\mathrm{Pic}}^0 J$, which generalizes the Poincaré line bundle for a smooth abelian surface.

Corollary 21. *Let D be an arbitrary curve of type* (2), (3)*, or* (4)*, and let $J := \overline{\mathrm{Pic}}^0 D$ be its compactified Jacobian, which is a degeneration of a principally polarized abelian surface. The universal sheaf $\iota^*\mathcal{U}$ described above induces an integral transform*

$$\Phi^{\iota^*\mathcal{U}}_{\overline{\mathrm{Pic}}^0 J \to J} : \mathcal{D}^b_{\mathrm{coh}}\big(\overline{\mathrm{Pic}}^0 J\big) \to \mathcal{D}^b_{\mathrm{coh}}(J)$$

which is an equivalence of triangulated categories.

Proof. We know that the universal sheaf $\mathcal{U}$ on the direct product $Z^0 \times P$ can be constructed from a relative Poincaré sheaf on the fibre product $Z^0 \times_{\mathbb{P}^2} P$, which we will also denote by $\mathcal{U}$ (abusing notation). The Fourier–Mukai transform $\Phi^{\mathcal{U}}_{P \to Z^0}$ is the same, regardless of whether it is constructed using the universal sheaf on $Z^0 \times P$ or the relative Poincaré sheaf on $Z^0 \times_{\mathbb{P}^2} P$. Denote it simply by Φ, and write Φ_D for $\Phi^{\iota^*\mathcal{U}}_{\overline{\mathrm{Pic}}^0 J \to J}$.

The corollary now follows from the faithful base change results proved by Kuznetsov in Section 2.6 of [31] (in particular, part (iv) of Proposition 2.43). In our case, we base change from $\mathbb{P}^2$

to the point corresponding to D, which allows us to specialize from the relative Fourier–Mukai transform Φ to Φ_D. Even when specializing to a smooth fibre, the proof that we get an equivalence for the fibres is somewhat involved. For singular fibres there are additional subtleties, which arise because one is dealing with sheaves on singular varieties. The reader may consult [31] for details. □

Corollary 22. *Let D be an arbitrary curve of type* (2), (3), *or* (4), *and let $J := \overline{\mathrm{Pic}}^0 D$ be its compactified Jacobian. Let L_1 and L_2 be two non-isomorphic torsion-free rank one sheaves of degree zero on J. Then*

$$\mathrm{Ext}^p_J(L_1, L_2) = 0$$

for all p.

Proof. This follows from Corollary 21, since L_1 and L_2 are the Fourier–Mukai transforms of two skyscraper sheaves supported at distinct points of $\overline{\mathrm{Pic}}^0 J$, and Φ_D preserves the $\mathrm{Ext}^\bullet$-pairing. However, we can give a direct proof using Theorem 19, which says that $\Phi^{\mathcal{U}}_{P\to Z^0}$ is an equivalence of triangulated categories. By Bridgeland's criterion, Theorem 1, it follows that we must have

$$\mathrm{Ext}^i_{Z^0}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0$$

for all integers i and all $m_1 \neq m_2 \in P$. In particular, suppose that m_1 and m_2 are points in the singular fibre $\overline{\mathrm{Pic}}^0 J$ corresponding to L_1 and L_2, so that $\mathcal{U}_{m_1} = \iota_* L_1$ and $\mathcal{U}_{m_2} = \iota_* L_2$.

By the remark preceding Lemma 14, the fibration $Z^0 \to \mathbb{P}^2$ is flat. Therefore we can pull-back the Koszul resolution of a point in $\mathbb{P}^2$ (*any* point) to get a resolution of the structure sheaf of the corresponding fibre. It follows that the spectral sequence in Section 7.2 of Bridgeland and Maciocia [10] exists also for singular fibres in our case, and thus we have

$$E_2^{p,q} := \mathrm{Ext}^p_J\big(L_1 \otimes \Lambda^q \mathcal{O}_J^{\oplus 2}, L_2\big) \Rightarrow \mathrm{Ext}^{p+q}_{Z^0}(\iota_* L_1, \iota_* L_2).$$

The vanishing of the right-hand side allows us to conclude that

$$E_2^{0,0} = \mathrm{Ext}^0_J(L_1, L_2) = \mathrm{Ext}^0_{Z^0}(\iota_* L_1, \iota_* L_2)$$

vanishes (this also follows since L_1 and L_2 are stable and not isomorphic). Now suppose that $\mathrm{Ext}^p_J(L_1, L_2)$ vanishes for all $p \leqslant k$. Then $E_2^{p,q}$ vanishes for $p \leqslant k$ and for all q. Therefore

$$E_2^{k+1,0} = \mathrm{Ext}^{k+1}_J(L_1, L_2) = \mathrm{Ext}^{k+1}_{Z^0}(\iota_* L_1, \iota_* L_2)$$

also vanishes. By induction

$$\mathrm{Ext}^p_J(L_1, L_2)$$

vanishes for all p. Note that since J is not smooth, L_1 and L_2 need not have finite projective resolutions. Thus we showed the vanishing for all p, rather than just $p \leqslant 2 = \dim J$. □

Remark. Note that J has trivial normal bundle in Z^0, and Z^0 has trivial canonical bundle, so by adjunction the canonical bundle $\mathcal{K}_J$ of J is also trivial. Also, the sheaves on J parametrized by $\overline{\mathrm{Pic}}^0\, J$ are clearly simple. One might expect that combining these observations with Corollary 22, we could obtain another proof of Corollary 21, using Bridgeland's criterion, Theorem 1. However, Bridgeland's result is for smooth varieties and it is not immediately clear how to generalize his proofs in [9] to singular varieties.[1]

Remark. Let us make one more remark about Theorem 19: although $P \cong Z^0$, it is important to distinguish these spaces in the sense that P should be regarded as the dual fibration of Z^0. In this paper, our four-folds Z^0 and Z^1 are fibred by principally polarized abelian surfaces, which are self-dual, which is why Z^0 happens to be isomorphic to its dual fibration.

However, there are holomorphic symplectic manifolds which are fibred by non-principally polarized abelian varieties. The generalized Kummer varieties provide examples (see Debarre [14]); indeed Proposition 5.3 of [45] shows that generalized Kummer varieties cannot be fibred by principally polarized abelian varieties. The possibility of constructing a Fourier–Mukai transform for generalized Kummer four-folds was briefly discussed in Section 5.4 of [46]; there remain many technical details to be resolved.

4.3. A twisted Fourier–Mukai transform

At the beginning of this section we made the observation that

$$\overline{\mathrm{Pic}}^0\big(Z^0/\mathbb{P}^2\big) \cong \overline{\mathrm{Pic}}^0\big(Z^1/\mathbb{P}^2\big).$$

Thus P is the dual fibration of both Z^0 and Z^1. In particular, P parametrizes stable sheaves on Z^1, so there exist local universal sheaves on $Z^1 \times P_i$ (for some cover $\{P_i\}$ of P). As in Section 2, there exists a gerbe $\beta \in \mathrm{H}^2(P, \mathcal{O}^*)$ which is the obstruction to the existence of a global universal sheaf on $Z^1 \times P$. Since Z^1 does not admit a section, we know from Proposition 9 that β is non-zero. The collection of local universal sheaves gives us a $\pi_P^*\beta$-twisted universal sheaf $\mathcal{U}$ on $Z^1 \times P$, where π_P is projection to P. We can therefore construct the integral transform with (twisted) kernel $\mathcal{U}$

$$\Phi^{\mathcal{U}}_{P\to Z^1} : \mathcal{D}^b_{\mathrm{coh}}\big(P, \beta^{-1}\big) \to \mathcal{D}^b_{\mathrm{coh}}\big(Z^1\big).$$

Theorem 23. *The functor $\Phi^{\mathcal{U}}_{P\to Z^1}$ is an equivalence of triangulated categories.*

Proof. We will apply Căldăraru's Theorem 4, which is the twisted version of Bridgeland's criterion, Theorem 1. Let $\mathcal{U}_m := \Phi^{\mathcal{U}}_{P\to Z^1}\mathcal{O}_m$ be the sheaf on Z^1 which the point $m \in P$ represents. We must show

[1] There have been some advances in the theory since the author first completed this article. In particular, Hernández Ruipérez, López Martín, and Sancho de Salas [23,24] have generalized Bridgeland's results to varieties with Gorenstein singularities. Our Corollary 21 could also be proved using their more general results about relative Fourier–Mukai transforms.

(1) for all $m \in P$, $\mathcal{U}_m \otimes \mathcal{K}_{Z^1} = \mathcal{U}_m$ and $\mathcal{U}_m$ is simple,
(2) for all integers i and all $m_1 \neq m_2 \in P$,

$$\mathrm{Ext}^i_{Z^1}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0.$$

The proof of condition (1) is the same as in the proof of Theorem 19. Namely, since $\mathcal{U}_m$ is stable, it is also simple. Since Z^1 is holomorphic symplectic, it has trivial canonical bundle $\mathcal{K}_{Z^1}$.

Regarding condition (2), first suppose that m_1 and m_2 lie in different fibres of P. Then as before $\mathcal{U}_{m_1}$ and $\mathcal{U}_{m_2}$ are sheaves supported on different (disjoint) fibres of Z^1. Therefore

$$\mathrm{Ext}^i_{Z^1}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0$$

for all i because all local $\mathcal{E}xt$ sheaves vanish.

Next suppose that m_1 and m_2 lie in the same fibre of P, which may be smooth or singular. Then $\mathcal{U}_{m_1}$ and $\mathcal{U}_{m_2}$ are of the form $\iota_* L_1$ and $\iota_* L_2$ respectively, where $\iota : Z^1_t \hookrightarrow Z^1$ is inclusion, and L_1 and L_2 are torsion-free rank one sheaves on Z^1_t of degree zero. As before we have the spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_{Z^1_t}\big(L_1 \otimes \Lambda^q \mathcal{O}^{\oplus 2}_{Z^1_t}, L_2\big) \Rightarrow \mathrm{Ext}^{p+q}_{Z^1}(\iota_* L_1, \iota_* L_2)$$

which exists for both smooth and singular fibres (see the comments in the proof of Corollary 22). If Z^1_t is smooth, and L_1 and L_2 are not isomorphic, then the spectral sequence vanishes as in the proof of Theorem 19. If Z^1_t is singular, then it is the compactified Jacobian of a curve of type (2), (3), or (4). Then Corollary 22 showed that

$$\mathrm{Ext}^p_{Z^1_t}(L_1, L_2)$$

vanishes for all p if L_1 and L_2 are not isomorphic. So once again, all terms in the spectral sequence vanish. It follows that

$$\mathrm{Ext}^i_{Z^1}(\mathcal{U}_{m_1}, \mathcal{U}_{m_2}) = 0$$

for all $i \in \mathbb{Z}$ and for all $m_1 \neq m_2 \in P$. This concludes the proof of condition (2), and of the theorem. □

Remark. Since the proof of Bridgeland and Maciocia's Theorem 2 only relies on local arguments, it can be generalized to the twisted case. This would lead to a direct proof of Theorem 23, without the need to first prove Theorem 19 and thereby obtain Corollary 22.

Remark. Recall that Z^1 is a torsor over $Z^0 \cong P$. Since Z^0 admits a section and Z^1 does not, we could regard Z^1 as being a 'twisted' version of the space Z^0. Thus we can paraphrase the theorem as saying that the derived category of twisted sheaves on the 'untwisted' space Z^0 is equivalent to the derived category of (untwisted) sheaves on the 'twisted' space Z^1.

4.4. Deformations of fibrations

In this final subsection we will show that Z^1 and Z^0 can be connected by a one-parameter family in their space of deformations, which only passes through Lagrangian fibrations. This follows by considering the subspace of $\mathrm{H}^2(P, \mathcal{O}^*)$ consisting of gerbes on P which arise from torsors over Z^0, and showing that it is connected.

We start with the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

on P. The cohomology of each of these sheaves on P can be computed using the projection $p_P : P \to \mathbb{P}^2$ and the Leray spectral sequence. Thus we have

$$E_2^{i,j}(\mathbb{Z}) := \mathrm{H}^i\big(\mathbb{P}^2, \mathbf{R}^j p_{P*}\mathbb{Z}\big) \Rightarrow \mathrm{H}^{i+j}(P, \mathbb{Z})$$

and since $\mathbf{R}^0 p_{P*}\mathbb{Z}_P \cong \mathbb{Z}_{\mathbb{P}^2}$ we can compute the bottom row of the left-hand side and obtain

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & & \\
\mathrm{H}^0\big(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathbb{Z}\big) & \mathrm{H}^1\big(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathbb{Z}\big) & \mathrm{H}^2\big(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathbb{Z}\big) & \dots & \\
\mathrm{H}^0\big(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathbb{Z}\big) & \mathrm{H}^1\big(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathbb{Z}\big) & \mathrm{H}^2\big(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathbb{Z}\big) & \dots & \\
\mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}.
\end{array}$$

We claim that the right-most term $\mathbb{Z}$ survives to $E_\infty^{4,0}(\mathbb{Z})$, i.e., the generator of this $\mathbb{Z}$ is not in the image of the higher derivations $d_2(\mathbb{Z})$, $d_3(\mathbb{Z})$, and $d_4(\mathbb{Z})$. By Theorem 7.2* on page 649 of Whitehead [50] we have a surjection

$$\mathrm{H}^4\big(\mathbb{P}^2, \mathbb{Z}\big) \cong E_2^{4,0}(\mathbb{Z}) \to E_\infty^{4,0}(\mathbb{Z})$$

and an injection

$$E_\infty^{4,0}(\mathbb{Z}) \to \mathrm{H}^4(P, \mathbb{Z})$$

whose composition is p_P^*. The generator η of $\mathrm{H}^4(\mathbb{P}^2, \mathbb{Z})$ is Poincaré dual to a point, so $p_P^*\eta \in \mathrm{H}^4(P, \mathbb{Z})$ is Poincaré dual to a fibre of $P \to \mathbb{P}^2$. In particular $p_P^*\eta$ is non-trivial, and the claim follows.

Next we have

$$E_2^{i,j}(\mathcal{O}) := \mathrm{H}^i\big(\mathbb{P}^2, \mathbf{R}^j p_{P*}\mathcal{O}\big) \Rightarrow \mathrm{H}^{i+j}(P, \mathcal{O}).$$

Matsushita [38] proved that $\mathbf{R}^j p_{P*}\mathcal{O}_P \cong \Omega^j_{\mathbb{P}^2}$ and therefore we can compute the left-hand side precisely, obtaining

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
0 & 0 & \mathbb{C} & \dots \\
0 & \mathbb{C} & 0 & \dots \\
\mathbb{C} & 0 & 0 & \dots
\end{array}$$

The spectral sequence degenerates at the E_2 term and gives

$$\mathrm{H}^k(P,\mathcal{O}) \cong \begin{cases} \mathbb{C} & k=0,2,4,\\ 0 & \text{otherwise.} \end{cases}$$

Finally we have

$$E_2^{i,j}(\mathcal{O}^*) := \mathrm{H}^i(\mathbb{P}^2, \mathbf{R}^j p_{P*}\mathcal{O}^*) \Rightarrow \mathrm{H}^{i+j}(P,\mathcal{O}^*).$$

In this case we know that $\mathbf{R}^0 p_{P*}\mathcal{O}_P^* \cong \mathcal{O}_{\mathbb{P}^2}^*$, and thus we can compute the bottom row of the left-hand side (using the exponential long exact sequence on $\mathbb{P}^2$), obtaining

$$\begin{array}{ccccc}
\vdots & \vdots & \vdots & & \\
\mathrm{H}^0(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathcal{O}^*) & \mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathcal{O}^*) & \mathrm{H}^2(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathcal{O}^*) & \ldots & \\
\mathrm{H}^0(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*) & \mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*) & \mathrm{H}^2(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*) & \ldots & \\
\mathbb{C}^* & \mathbb{Z} & 0 & \mathbb{Z} & \ldots.
\end{array}$$

Now Z^1 is a torsor over Z^0, corresponding to a gerbe $\beta \in \mathrm{H}^2(P,\mathcal{O}^*)$. Moreover, there is a cover $\{P_i\}$ of P obtained by pulling back a cover of $\mathbb{P}^2$, and β can be represented by line bundles $\mathcal{L}_{ij}$ on pair-wise intersections P_{ij}. Moreover, we can assume that these line bundles have degree zero on each fibre of $p_P : P \to \mathbb{P}^2$ (as we saw in the second half of the proof of Proposition 9). Conversely, suppose we are given a gerbe β' with these properties. Once again, the proof of Proposition 9 showed that we can construct a torsor $Z^{\beta'}$ over Z^0 (the resulting space $Z^{\beta'}$ might not be projective, but the construction still applies). Now a family of line bundles on fibres of P is a local section of $\mathbf{R}^1 p_{P*}\mathcal{O}^*$, and the degree is given by the coboundary map δ_1 of the long exact sequence of direct image sheaves

$$\cdots \to \mathbf{R}^1 p_{P*}\mathcal{O} \to \mathbf{R}^1 p_{P*}\mathcal{O}^* \xrightarrow{\delta_1} \mathbf{R}^2 p_{P*}\mathbb{Z} \to \mathbf{R}^2 p_{P*}\mathcal{O} \to \cdots.$$

Thus β corresponds to a cocycle in $\mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*)$ which moreover can be represented by local sections of

$$\ker\delta_1 \subset \mathbf{R}^1 p_{P*}\mathcal{O}^*.$$

Looking at the spectral sequence, we see that this cocycle must also lie in the kernel of

$$d_2(\mathcal{O}^*) : \mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*) \to \mathrm{H}^3(\mathbb{P}^2, \mathbf{R}^0 p_{P*}\mathcal{O}^*) \cong \mathbb{Z}$$

since it survives to give a class in $\mathrm{H}^2(P, \mathcal{O}^*)$. The next lemma shows that this makes the condition on the degree redundant.

Lemma 24. *Suppose that α lies in the kernel of*

$$d_2(\mathcal{O}^*) : \mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*) \to \mathrm{H}^3(\mathbb{P}^2, \mathbf{R}^0 p_{P*}\mathcal{O}^*) \cong \mathbb{Z}.$$

Then α can be represented by line bundles $\mathcal{L}_{ij}$ on pair-wise intersections P_{ij} which have degree zero on each fibre of $p_P : P \to \mathbb{P}^2$.

Proof. By the functoriality of spectral sequences, the coboundary maps of the long exact sequence of direct image sheaves

$$\cdots \to \mathbf{R}^0 p_{P*}\mathcal{O}^* \xrightarrow{\delta_0} \mathbf{R}^1 p_{P*}\mathbb{Z} \to \mathbf{R}^1 p_{P*}\mathcal{O} \to \mathbf{R}^1 p_{P*}\mathcal{O}^* \xrightarrow{\delta_1} \mathbf{R}^2 p_{P*}\mathbb{Z} \to \cdots$$

induce maps which commute with the derivations of the spectral sequences. Thus we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*) \arrow[r, "d_2(\mathcal{O}^*)"] \arrow[d, "\mathrm{H}^1(\delta_1)"] & \mathrm{H}^3(\mathbb{P}^2, \mathbf{R}^0 p_{P*}\mathcal{O}^*) \arrow[d, "\mathrm{H}^3(\delta_0)"] \\
\mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathbb{Z}) \arrow[r, "d_2(\mathbb{Z})"] & \mathrm{H}^3(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathbb{Z}).
\end{tikzcd}$$

Since α is in the kernel of $d_2(\mathcal{O}^*)$, $\mathrm{H}^1(\delta_1)\alpha$ must lie in the kernel of $d_2(\mathbb{Z})$. Now as observed above, $\mathrm{H}^4(\mathbb{P}^2, \mathbf{R}^0 p_{P^*}\mathbb{Z}) \cong \mathbb{Z}$ must survive to give a class in $\mathrm{H}^4(P, \mathbb{Z})$, and thus the map

$$d_3(\mathbb{Z})|_{\ker d_2(\mathbb{Z})} : \ker d_2(\mathbb{Z}) \subset \mathrm{H}^1\big(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathbb{Z}\big) \to \mathrm{H}^4\big(\mathbb{P}^2, \mathbf{R}^0 p_{P*}\mathbb{Z}\big)$$

is trivial. This means that $\mathrm{H}^1(\delta_1)\alpha$ also lies in the kernel of $d_3(\mathbb{Z})$, and thus it survives to give a class in $\mathrm{H}^3(P, \mathbb{Z})$. However, P is a deformation of the Hilbert scheme of two points on a K3 surface, and thus $\mathrm{H}^3(P, \mathbb{Z}) = 0$. We conclude that $\mathrm{H}^1(\delta_1)\alpha = 0$.

So α is given by local sections α_{ij} of $\mathbf{R}^1 p_{P*}\mathcal{O}^*$ (i.e., the line bundles $\mathcal{L}_{ij}$) over the pair-wise intersections $p_P(P_{ij})$, such that the cocycle $\{\delta_1\alpha_{ij}\}$ is actually a coboundary. So there exist local sections γ_i of $\mathbf{R}^2 p_{P*}\mathbb{Z}$ over $p_P(P_i)$ such that

$$\delta_1\alpha_{ij} = \gamma_i - \gamma_j.$$

Consider the image of γ_i under the map

$$\mathbf{R}^2 p_{P*}\iota : \mathbf{R}^2 p_{P*}\mathbb{Z} \to \mathbf{R}^2 p_{P*}\mathcal{O}$$

induced by the inclusion $\iota : \mathbb{Z} \to \mathcal{O}$. Observe that

$$\mathbf{R}^2 p_{P*}\iota\gamma_i - \mathbf{R}^2 p_{P*}\iota\gamma_j = \mathbf{R}^2 p_{P*}\iota(\delta_1\alpha_{ij})$$

vanishes, by the exactness of

$$\cdots \to \mathbf{R}^1 p_{P*}\mathcal{O}^* \xrightarrow{\delta_1} \mathbf{R}^2 p_{P*}\mathbb{Z} \xrightarrow{\mathbf{R}^2 p_{P*}\iota} \mathbf{R}^2 p_{P*}\mathcal{O} \to \cdots$$

Therefore the $\mathbf{R}^2 p_{P*}\iota\gamma_i$ agree on overlaps and can be patched together to give a global section of $\mathbf{R}^2 p_{P*}\mathcal{O}$. However, Matsushita proved in [38] that $\mathbf{R}^2 p_{P*}\mathcal{O} \cong \Omega^2_{\mathbb{P}^2}$, which has no global sections, and therefore $\mathbf{R}^2 p_{P*}\iota\gamma_i$ vanishes for all i.

Using the exactness of the long exact sequence of direct image sheaves once again, we conclude that there exist local sections ϵ_i of $\mathbf{R}^1 p_{P*}\mathcal{O}^*$ over $p_P(P_i)$ such that

$$\gamma_i = \delta_1\epsilon_i.$$

Now define

$$\alpha'_{ij} := \alpha_{ij} - \epsilon_i + \epsilon_j.$$

These are local sections of $\mathbf{R}^1 p_{P*}\mathcal{O}^*$ over $p_P(P_{ij})$ (we have written the group action additively, but if one wants to think of these as families of line bundles on fibres then the group action is just tensor product). Moreover the collection $\{\alpha'_{ij}\}$ represents the same cohomology class in $\mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*)$ as α, though now we have

$$\begin{aligned}\delta_1\alpha' &= \delta_1\alpha_{ij} - \delta_1\epsilon_i + \delta_1\epsilon_j \\ &= \gamma_i - \gamma_j - \gamma_i + \gamma_j \\ &= 0.\end{aligned}$$

So α'_{ij} corresponds to a line bundle $\mathcal{L}'_{ij}$ on P_{ij} which has degree zero on each fibre of $p_P : P \to \mathbb{P}^2$. This completes the proof. □

Finally we prove the following result.

Theorem 25. *There is a one-parameter family of Lagrangian fibrations Z^t over $\mathbb{P}^2$, connecting Z^0 and Z^1 in their moduli space of deformations. Each fibration $Z^t \to \mathbb{P}^2$ is a torsor over Z^0, and it corresponds to a gerbe $\beta_t \in \mathrm{H}^2(P, \mathcal{O}^*)$.*

Proof. Let $\beta \in \mathrm{H}^2(P, \mathcal{O}^*)$ be the gerbe corresponding to Z^1, and α the corresponding class in $\mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*)$. Then α survives the spectral sequence to give the class β. Part of the exponential long exact sequence on P looks like

$$\cdots \to \mathbb{C} \xrightarrow{\mathrm{H}^2(\exp)} \mathrm{H}^2\big(P, \mathcal{O}^*\big) \to \mathrm{H}^3(P, \mathbb{Z}) \to \cdots$$

since $\mathrm{H}^2(P, \mathcal{O}) \cong \mathbb{C}$. We know that β can be represented by line bundles $\mathcal{L}_{ij}$ on P_{ij} which have degree zero on each fibre of $p_P : P \to \mathbb{P}^2$, which implies that $\mathrm{H}^1(\delta_1)\alpha = 0$. Once again by functoriality of the spectral sequences, β must map to zero in $\mathrm{H}^3(P, \mathbb{Z})$ (the image of β in $\mathrm{H}^3(P, \mathbb{Z})$ will come from the image $\mathrm{H}^1(\delta_1)\alpha$ of α in $\mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^2 p_{P*}\mathbb{Z})$, via the spectral sequence for $\mathbb{Z}$). Of course one could simply observe that $\mathrm{H}^3(P, \mathbb{Z})$ itself vanishes, but we do not actually need this fact.

By exactness, β is the image under $\mathrm{H}^2(\exp)$ of an element in $\mathrm{H}^2(P, \mathcal{O}) \cong \mathbb{C}$. Likewise, α must be the image of a class

$$\kappa \in \mathrm{H}^1\big(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}\big) \cong \mathbb{C}$$

under the map $\mathrm{H}^1(\mathbf{R}^1 p_{P*}\exp)$ coming from the exponential $\exp : \mathcal{O} \to \mathcal{O}^*$. Next let us define $\alpha_t \in \mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*)$ to be the image of $t\kappa \in \mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O})$ under $\mathrm{H}^1(\mathbf{R}^1 p_{P*}\exp)$, with t a real number in the interval [0, 1]. Since

$$\begin{array}{ccc}
\mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}) & \xrightarrow{d_2(\mathcal{O})} & \mathrm{H}^3(\mathbb{P}^2, \mathbf{R}^0 p_{P*}\mathcal{O}) = 0 \\
\downarrow{\scriptstyle \mathrm{H}^1(\mathbf{R}^1 p_{P*}\exp)} & & \downarrow{\scriptstyle \mathrm{H}^3(\mathbf{R}^0 p_{P*}\exp)} \\
\mathrm{H}^1(\mathbb{P}^2, \mathbf{R}^1 p_{P*}\mathcal{O}^*) & \xrightarrow{d_2(\mathcal{O}^*)} & \mathrm{H}^3(\mathbb{P}^2, \mathbf{R}^0 p_{P*}\mathcal{O}^*)
\end{array}$$

commutes, α_t must lie in the kernel of $d_2(\mathcal{O}^*)$. Thus not only does it survive to give a gerbe $\beta_t \in \mathrm{H}^2(P, \mathcal{O}^*)$, but by Lemma 24 it can be represented by line bundles $\mathcal{L}_{ij}$ on P_{ij} which have degree zero on each fibre of $p_P : P \to \mathbb{P}^2$. This is precisely what is required to construct the torsor Z^t over Z^0, as in the proof of Proposition 9. □

Remark. Note that in the proof of Lemma 24, the vanishing of $\mathrm{H}^3(P, \mathbb{Z})$ was used to show that $\mathrm{H}^1(\delta_1)\alpha = 0$. However, in the proof of Theorem 25 we already know that $\mathrm{H}^1(\delta_1)\alpha_t = 0$, so the argument would work even if $\mathrm{H}^3(P, \mathbb{Z})$ did not vanish. So even if $\mathrm{H}^2(P, \mathcal{O}^*)$ is not connected, the gerbes on P which arise from torsors over Z^0 form a connected subspace. This is significant because for the generalized Kummer four-fold K_4 we have $\mathrm{H}^3(K_4, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 8}$. In Section 5.4 of [46] the author suggested one might be able to find new deformation classes of holomorphic symplectic four-folds by constructing Lagrangian fibrations from gerbes in different connected components of $\mathrm{H}^2(K_4, \mathcal{O}^*)$. Unfortunately the above argument suggests that this will not work.

Remark. Let $\mathrm{Def}(Z^0)$ be the Kuranishi space parametrizing deformations of Z^0 as a complex manifold. Matsushita [38] proved that there is a subspace $\Delta \subset \mathrm{Def}(Z^0)$ of codimension one parametrizing deformations of Z^0 which are Lagrangian fibrations, and in [47] the author proved that there is a subspace $\Delta' \subset \Delta \subset \mathrm{Def}(Z^0)$ of codimension one in Δ (and hence codimension two in $\mathrm{Def}(Z^0)$) parametrizing deformations of Z^0 which are Lagrangian fibrations with global sections. The one-parameter family described in Theorem 25 is consistent with these results; the generic Z^t will not admit a global section (since Z^1 does not) and hence the one-parameter family can be regarded as a deformation inside Δ but transverse to Δ'.

Remark. Finally, we observe that the derived category $\mathcal{D}^b_{\mathrm{coh}}(Z^t)$ of each space Z^t is equivalent to the derived category $\mathcal{D}^b_{\mathrm{coh}}(P, \beta_t^{-1})$ of β_t^{-1}-twisted sheaves on P. This can be proved in the same way as Theorem 23. Namely, P will be the compactified relative Picard scheme of Z^t and there will be a twisted universal sheaf inducing a twisted Fourier–Mukai transform from $\mathcal{D}^b_{\mathrm{coh}}(Z^t)$ to $\mathcal{D}^b_{\mathrm{coh}}(P, \beta_t^{-1})$. Although Z^t could be non-projective (i.e., when β_t is not a torsion element in $\mathrm{H}^2(P, \mathcal{O}^*)$) this will not present any problems since Căldăraru's results only require Z^t and P to be proper and smooth schemes over $\mathbb{C}$ or compact complex manifolds. These equivalences suggests that, as far as the derived category can detect, deforming the space is the same as deforming the gerbe while keeping the space fixed.

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